Group acting on Λ trees

Olga Kharlampovich
(McGill University)

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This talk is based on joint results with A. Myasnikov and D. Serbin.

The starting point

Theorem . A group G is free if and only if it acts freely on a tree.

Free action = no inversion of edges and stabilizers of vertices are trivial.

Ordered abelian groups

$\Lambda =$ an ordered abelian group (any $a, b \in \Lambda$ are comparable and for any $c \in \Lambda$: $a \leq b \Rightarrow a + c \leq b + c$).

Examples:

Archimedean case:

$\Lambda = \mathbb{R}$, $\Lambda = \mathbb{Z}$ with the usual order.

Non-Archimedean case:

$\Lambda = \mathbb{Z}^2$ with the right lexicographic order:

$$(a, b) < (c, d) \iff b < d \text{ or } b = d \text{ and } a < c.$$

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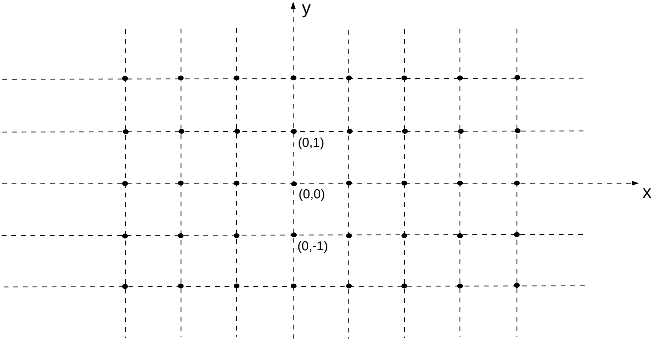
$\mathbb{Z}^2$ with the right-lex ordering



One-dimensional picture

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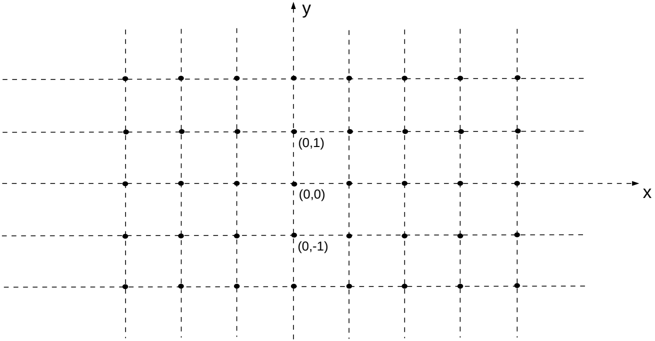

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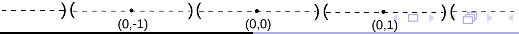

Λ-trees

Morgan and Shalen (1985) defined Λ-trees:
A Λ-tree is a metric space (X, p) (where $p : X \times X \to \Lambda$) which satisfies the following properties:

1) (X, p) is geodesic,
2) if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment,
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> Find the group theoretic information carried by an action on a Λ-tree.

Generalize Bass-Serre theory (for actions on $\mathbb{Z}$-trees) to actions on arbitrary Λ-trees.

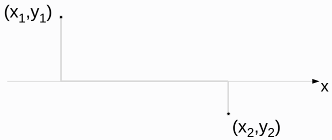
Examples for $\Lambda = \mathbb{R}$

$X = \mathbb{R}$ with usual metric.

A geometric realization of a simplicial tree.

$X = \mathbb{R}^2$ with metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$



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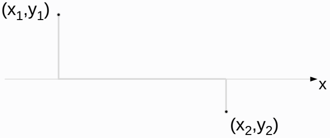

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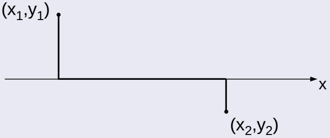

Finitely generated $\mathbb{R}$-free groups

Rips' Theorem [Rips, 1991 - not published]

A f.g. group acts freely on $\mathbb{R}$-tree if and only if it is a free product of surface groups (except for the non-orientable surfaces of genus 1,2, 3) and free abelian groups of finite rank.

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Properties

Some properties of groups acting freely on Λ-trees (Λ-free groups)

1. The class of Λ-free groups is closed under taking subgroups and free products.
2. Λ-free groups are torsion-free.
3. Λ-free groups have the CSA-property (maximal abelian subgroups are malnormal).
4. Commutativity is a transitive relation on the set of non-trivial elements.
5. Any two-generator subgroup of a Λ-free group is either free or free abelian.

The Fundamental Problem

The following is a principal step in the Alperin-Bass' program:

Open Problem [Rips, Bass]

Describe finitely generated groups acting freely on Λ-trees.

Here "describe" means "describe in the standard group-theoretic terms".

We solved this problem for finitely presented groups.
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Non-Archimedean actions

Theorem (H.Bass, 1991)

A finitely generated $(\Lambda \oplus \mathbb{Z})$-free group is the fundamental group of a finite graph of groups with properties:

- vertex groups are Λ-free,
- edge groups are maximal abelian (in the vertex groups),
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Actions on $\mathbb{R}^n$-trees

Theorem [Guirardel, 2003]

A f.g. freely indecomposable $\mathbb{R}^n$-free group is isomorphic to the fundamental group of a finite graph of groups, where each vertex group is f.g. $\mathbb{R}^{n-1}$-free, and each edge group is cyclic.

However, the converse is not true.

Corollary A f.g. $\mathbb{R}^n$-free group is hyperbolic relative to abelian subgroups.

Notice, that $\mathbb{Z}^n$-free groups are $\mathbb{R}^n$-free.

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$\mathbb{Z}^n$-free groups

Theorem [Kharlampovich, Miasnikov, Remeslennikov, 96]

Finitely generated fully residually free groups are $\mathbb{Z}^n$-free.

Theorem [Martino and Rourke, 2005]

Let G_1 and G_2 be $\mathbb{Z}^n$-free groups. Then the amalgamated product $G_1 *_C G_2$ is $\mathbb{Z}^m$-free for some $m \in \mathbb{N}$, provided C is cyclic and maximal abelian in both factors.

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Examples of $\mathbb{Z}^n$-free groups:

$\mathbb{R}$-free groups,

$\langle\, x_1, x_2, x_3 \mid x_1^2\ x_2^2\ x_3^2 = 1 \,\rangle$ is $\mathbb{Z}^2$-free (but is neither $\mathbb{R}$-free, nor fully residually free).

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From actions to length functions

Let G be a group acting on a Λ-tree (X, d). Fix a point $x_0 \in X$ and consider a function $l : G \longrightarrow \Lambda$ defined by

$$l(g) = d(x_0 \ , \ gx_0)$$

l is called a based length function on G with respect to x_0, or a Lyndon length function.

l is free if the underlying action is free.

Example. In a free group F, the function $f \to |f|$ is a free $\mathbb{Z}$-valued (Lyndon) length function.

Regular action

Definition

Let G act on a Λ-tree Γ. The action is regular with respect to $x \in \Gamma$ if for any $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$. We say that G is complete.

Comments

Let G act on a Λ-tree (Γ, d). Then the action of G is regular with respect to $x \in \Gamma$ if and only if the length function $l_x : G \to \Lambda$ based at x is regular.

Let G act minimally on a Λ-tree Γ. If the action of G is regular with respect to $x \in \Gamma$ then all branch points of Γ are G-equivalent.

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Main Theorems

Theorem

[KMS] Any f.p. complete Λ-free group G can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian and length isomorphic as subgroups of Λ.

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[KMS] Any finitely presented Λ-free group $\tilde{G}$ is $\mathbb{R}^n$-free, where $\mathbb{R}^n$ is ordered lexicographically for an appropriate $n \in \mathbb{N}$.

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Length functions

Length functions were introduced by **Lyndon (1963)**.

Let G be a group. A function $l : G \to \Lambda$ is called a length function on G if

(L1) $\forall\, g \in G: \ l(g) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\, g \in G: \ l(g) = l(g^{-1})$,

(L3) the triple $\{c(g,f), c(g,h), c(f,h)\}$ is isosceles for all $g,f,h \in G$, where $c(f,g)$ is the Gromov's product:

$$c(g,f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

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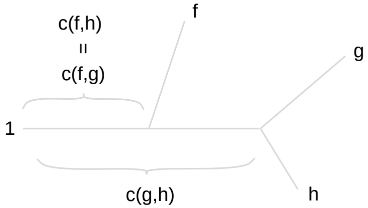
Examples of length functions

In a free group F, the function $f \to |f|$ is a $\mathbb{Z}$-valued length function.

For $f, g, h \in F$ we have



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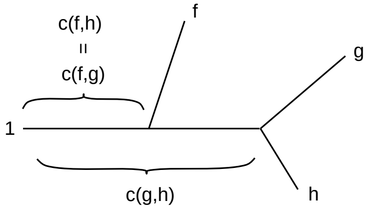

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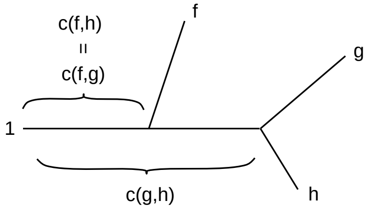

Free length functions

A length function $l : G \to \Lambda$ is free if $l(g^2) > l(g)$ for every non-trivial $g \in G$.

Chiswell's Theorem

Theorem [Chiswell]

Let $L : G \to \Lambda$ be a Lyndon length function on a group G. Then there exists a Λ-tree (X, d), $x \in X$, and an isometric action of G on X such that $L(g) = d(x, gx)$ for all $g \in G$.

Notice that $L(g) = d(x, gx)$ is free iff the action of G on X is free.

This gives another approach to free actions on Λ-trees.

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Infinite words

Let Λ be a discretely ordered abelian group with a minimal positive element 1_Λ and $X = \{x_i \mid i \in I\}$ be a set.

An **Λ-word** is a function

$$w : [1_\Lambda, \alpha] \to X^{\pm}, \quad \alpha \in \Lambda.$$

$|w| = \alpha$ is called the length of w.

w is **reduced** $\iff$ no subwords xx^{-1}, $x^{-1}x$ $(x \in X)$.

$R(\Lambda, X) =$ the set of all reduced Λ-words.

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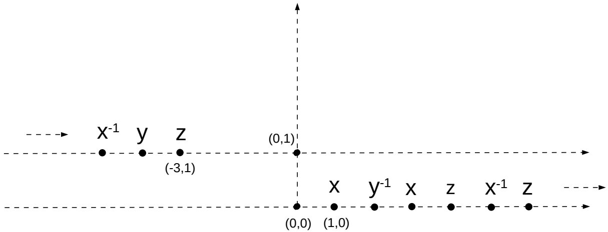
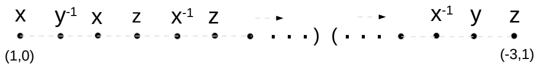
Example.

Let $X = \{x, y, z\}$, $\Lambda = \mathbb{Z}^2$



In "linear" notation



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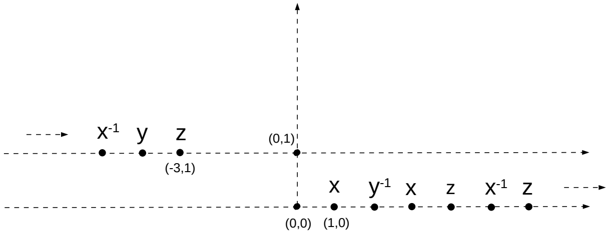

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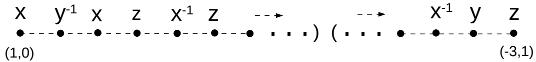
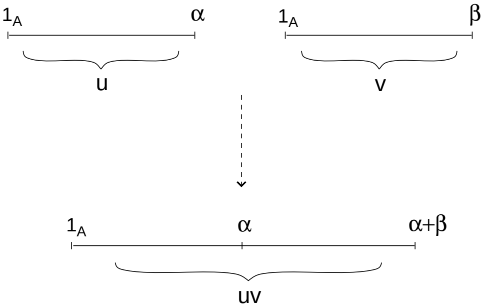
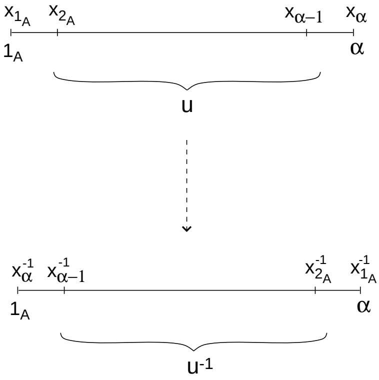


Concatenation of Λ-words:

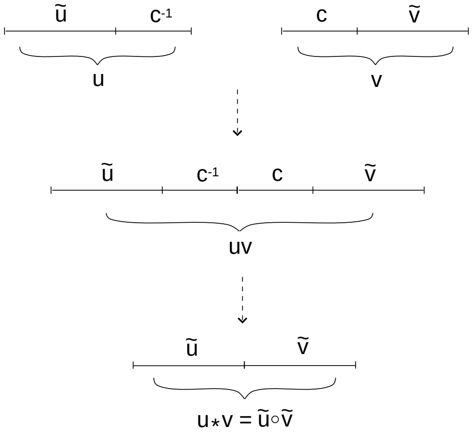


We write $u \circ v$ instead of uv in the case when uv is reduced.

Inversion of Λ-words:



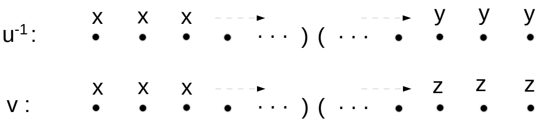
Multiplication of Λ-words:



The partial group $R(\Lambda, X)$

The multiplication on $R(\Lambda, X)$ is partial, it is not everywhere defined!

Example. $u, v \in R(\mathbb{Z}^2, X)$



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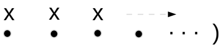

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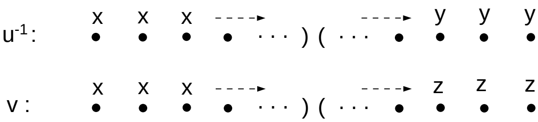

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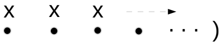

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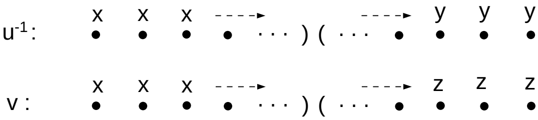

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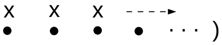

Cyclic decompositions

$v \in R(\Lambda, X)$ is **cyclically reduced** if $v(1_A)^{-1} \neq v(|v|)$.

$v \in R(\Lambda, X)$ admits a **cyclic decomposition** if

$$v = c^{-1} \circ u \circ c,$$

where $c, u \in R(A, \Lambda)$ and u is cyclically reduced.

Denote by $CDR(A, \Lambda)$ the set of all words from $R(\Lambda, X)$ which admit a cyclic decomposition.

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From Non-Archimedean words - to length functions

Theorem [Myasnikov-Remeslennikov-Serbin, 2003]

Let Λ be a discretely ordered abelian group and X a set. If G is a subgroup of $CDR(\Lambda, X)$ then the function $L_G : G \to \Lambda$, defined by $L_G(g) = |g|$, is a free Lyndon length function.

Corollary.

To show that a group G acts on a Λ-tree - embed G into $CDR(\Lambda, X)$.

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From Length functions - to Non-Archimedean words

Theorem [Chiswell], 2004

Let Λ be a discretely ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists an embedding $\phi : G \to CDR(\Lambda, X)$ such that $|\phi(g)| = L(g)$ for every $g \in G$.

Corollary. Let Λ be an arbitrary ordered abelian group. If $L : G \to \Lambda$ is a free Lyndon length function on a group G then there exists a length preserving embedding $\phi : G \to CDR(\Lambda', X)$, where $\Lambda' = \Lambda \oplus \mathbb{Z}$ with the lex order.

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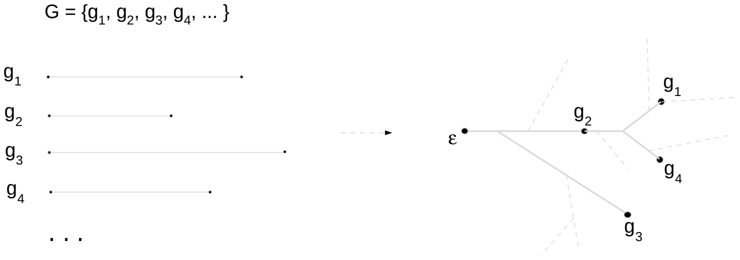
From Non-Archimedean words - to free actions

Infinite words $\Longrightarrow$ Length functions $\Longrightarrow$ Free actions

Shortcut

If $G \hookrightarrow CDR(\Lambda, X)$ then G acts by isometries on the canonical Λ-tree $\Gamma(G)$ labeled by letters from $X^{\pm}$.



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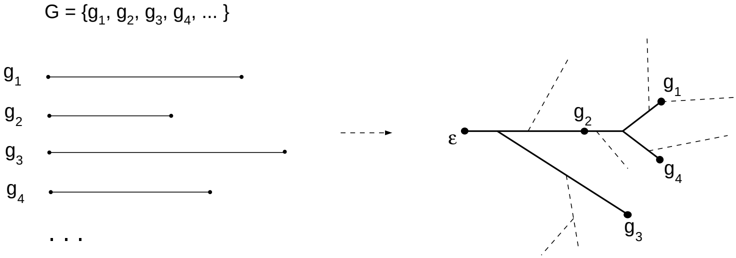

Regular free actions

A length function $l : G \to A$ is called *regular* if it satisfies the *regularity* axiom:

(L6) $\forall\, g, f \in G,\ \exists\, u, g_1, f_1 \in G$:

$$g = u \circ g_1 \ \&\ f = u \circ f_1 \ \&\ l(u) = c(g, f).$$

Complete subgroups

Let $G \leq CDR(\Lambda, X)$ be a group of infinite words.

Complete subgroups

$G \leq CDR(\Lambda, X)$ is complete if G contains the common initial segment $c(g, h)$ for every pair of elements $g, h \in G$.

Regular length functions

A Lyndon length function $L : G \to \Lambda$ is regular if there exists a length preserving embedding $G \to CDR(\Lambda, X)$ onto a complete subgroup.

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Complete subgroups

Example. Let $F(x, y)$ be a free group and $H = \langle x^2y^2,\ xy\rangle$ be its subgroup.

F has natural free $\mathbb{Z}$-valued length function $l_F : f \to |f|$. Hence, l_F induces a length function l_H on H.

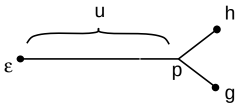
l_F is regular, but l_H is not

Take $g = xy^{-1}x^{-2},\ h = xy^{-1}x^{-1}y$ in F. Then

$$g, h \in H, \quad \text{but} \quad com(g, h) = xy^{-1}x^{-1} \notin H.$$

Branch points and completeness

A vertex $p \in \Gamma(G)$ is a branch point if it is the terminal endpoint of the common initial segment $u = com(g, h)$ of $g, h \in G$.



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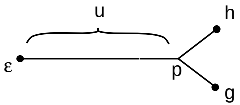

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Every finitely generated $\mathbb{Z}^n$-free group is a subgroup of a f.g. complete $\mathbb{Z}^n$-free group.

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Every finitely generated Λ-free group is $\mathbb{Z}^n$-free.

How probable it is?

Theorem [Kharlampovich, Myasnikov, Serbin]

Every finitely generated $\mathbb{R}$-free group is $\mathbb{Z}^2$-free for some $m \geq n$.

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Elimination process

Elimination Process (EP) is a dynamical (rewriting) process of a certain type that transforms formal systems of equations in groups or semigroups (or band complexes, or foliated 2-complexes, or partial isometries of multi-intervals) .
Makanin (1982): Initial version of EP.
Makanin's EP gives a decision algorithm to verify consistency of a given system of equations - decidability of the Diophantine problem over free groups.

Razborov's process

Razborov (1987): developed EP much further.
Razborov's EP produces **all solutions** of a given system in F.
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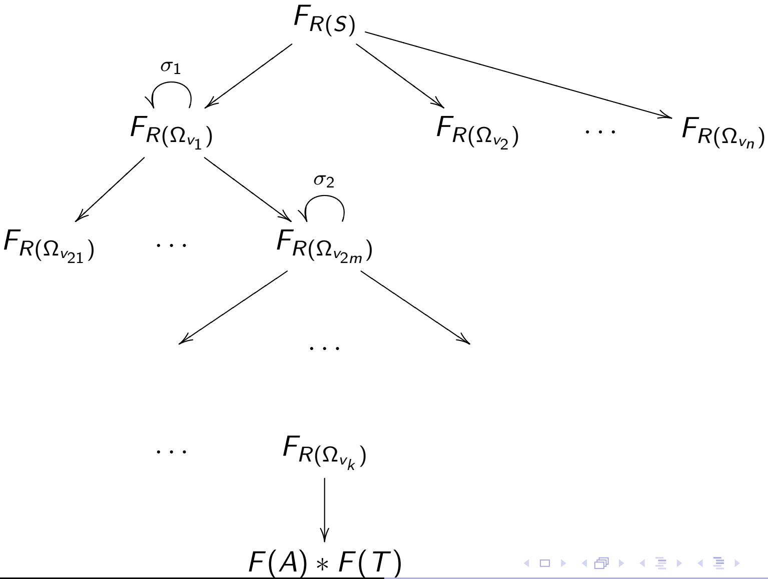
$F_{R(S)}$
σ_1
$F_{R(\Omega_{v_1})}$
$F_{R(\Omega_{v_2})}$
. . .
$F_{R(\Omega_{v_n})}$
σ_2
$F_{R(\Omega_{v_{21}})}$
. . .
$F_{R(\Omega_{v_{2m}})}$
. . .
. . .
$F_{R(\Omega_{v_k})}$
$F(A) * F(T)$

Kharlampovich - Myasnikov (1998):

Refined Razborov's process.
Effective description of solutions of equations in free (and fully residually free) groups in terms of very particular **triangular systems** of equations.
Resembles the classical elimination theory for polynomials.

Elimination process and splittings

A **splitting** of G is a representation of G as the fundamental groups of a graph of groups.
A splitting is **cyclic (abelian)** if all the edge groups are cyclic (abelian).

Elementary splittings:

$G = A *_C B, \quad G = A*_C = \langle A, t \mid t^{-1}Ct = C' \rangle,$

Free splittings:

$G = A * B$

Elimination processes and free actions

Infinite branches of an elimination process correspond precisely to the standard types of free actions:
linear case $\Longleftrightarrow$ **thin (or Levitt)** type
the quadratic case $\Longleftrightarrow$ **surface type (or interval exchange)**,
periodic structures $\Longleftrightarrow$ **toral (or axial)** type.

Bestvina-Feighn's elimination process

A powerful variation of the Makanin-Razborov's process for $\mathbb{R}$-actions.
Can be viewed as an asymptotic (limit) version of MR process.
Much simpler in applications but not algorithmic.

KM elimination process for $\mathbb{Z}^n$ actions

To solve equations in fully residually free groups we designed a variation of the elimination process for $\mathbb{Z}^n$ actions.

It **effectively** describes solution sets of finite systems of equations in $\mathbb{Z}^n$-groups in terms of **Triangular quasi-quadratic systems** (as in the case of fully residually free groups).

Non-standard version of Rip's machine

Kh., Myasnikov, and Serbin designed an elimination process for arbitrary non-Archimedean actions, i.e, free actions on Λ-trees.

This can be viewed as a **non-Archimedean (non-standard)** discrete, effective version of the original MR process.

Sketch of the proof of the theorem about Λ-free f.p. groups

Let G have a regular free length function in Λ.

Fix an embedding of G into $CDR(\Lambda, X)$ and construct a cancellation tree for each relation of G.

Sketch of the proof

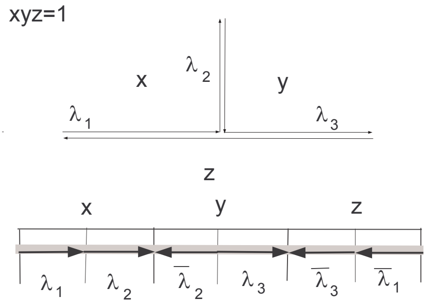

Figure: From the cancellation tree for the relation $xyz = 1$ to the generalized equation $(x = \lambda_1 \circ \lambda_2,\ y = \lambda_2^{-1} \circ \lambda_3,\ z = \lambda_3^{-1} \circ \lambda_1^{-1})$.

Sketch of the proof

Infinite branches of an elimination process correspond to abelian splittings of G:

linear case $\Longleftrightarrow$ **splitting as a free product.**

the quadratic case $\Longleftrightarrow$ **QH-subgroup**,

periodic structures $\Longleftrightarrow$ **abelian** vertex group or splitting as an HNN with abelian edge group.

After obtaining a splitting we apply EP to the vertex groups. We build the Delzant-Potyagailo hierarchy.

Sketch of the proof

A family $\mathcal{C}$ of subgroups of a torsion-free group G is called *elementary* if

(a) $\mathcal{C}$ is closed under taking subgroups and conjugation,
(b) every $C \in \mathcal{C}$ is contained in a maximal subgroup $\overline{C} \in \mathcal{C}$,
(c) every $C \in \mathcal{C}$ is small (does not contain F_2 as a subgroup),
(d) all maximal subgroups from $\mathcal{C}$ are malnormal.

G admits a *hierarchy* over $\mathcal{C}$ if the process of decomposing G into an amalgamated product or an HNN-extension over a subgroup from $\mathcal{C}$, then decomposing factors of G into amalgamated products and/or HNN-extensions over a subgroup from $\mathcal{C}$ etc. eventually stops.

Theorem (Delzant - Potyagailo (2001)). If G is a finitely presented group without 2-torsion and $\mathcal{C}$ is a family of elementary subgroups of G then G admits a hierarchy over $\mathcal{C}$.

Corollary. If G is a f.p. Λ-free group then G admits a hierarchy

Hyperbolic length functions Let G be a group, Λ an ordereg abelian group. A function $l : G \to \Lambda$ is called a δ-hyperbolic length function on G if

(L1) $\forall\, g \in G : \; l(g) \geqslant 0$ and $l(1) = 0$,

(L2) $\forall\, g \in G : \; l(g) = l(g^{-1})$,

(L3) $\forall\, g, h \in G : \; l(gh) \leq l(g) + l(h)$,

(L4) $\forall\, f, g, h \in G : \; c(f, g) \geq min\{c(f, h), c(g, h)\} - \delta$, where $c(f, g)$ is the Gromov's product:

$$c(g, f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f)).$$

A δ-hyperbolic length function is called complete if $\forall g \in G$, and $\alpha \leq l(g)$ there is $u \in G$ such that $g = u \circ g_1$, where $l(u) = \alpha$.

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Free hyperbolic length functions A δ -hyperbolic length function is called free (δ-free) if
$\forall g \in G: \ g \neq 1 \rightarrow l(g^2) > l(g) \ (resp., l(g^2) > l(g) - c(\delta)).$

Problem

Find the structure of f.g. groups with δ-hyperbolic, δ-regular, δ-free length function, in $\mathbb{Z}^n$, where $l(\delta)$ is in the smallest component of $\mathbb{Z}^n$.

A.P. Grecianu (McGill) obtained first results in this direction.