Algebraic Geometry for Groups

Olga Kharlampovich
(McGill University)

Cornell, 2011

Outline

In this talk I will discuss some new research areas, methods, and results which appeared in group theory in connection to solutions of Tarski problems about first order theory of free groups. I will also discuss elemenary classification questions and f.g. groups universally equivalent to free groups and elementary equivalent to free groups.

New areas: Algebraic geometry over groups, limit groups, groups acting freely on non-Archimedean trees, free actions on Λ-hyperbolic spaces, algebraic theory of equations in groups.

New methods: Elimination Processes (dynamical processes, transformations similar to interval exchange), regular actions, non-Archimedean words and presentations, Lyndon's completions.

New interesting open problems.

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Tarski's Problems

In 1945 Alfred Tarski posed the following problems.

Do the elementary theories of free non-abelian groups F_n and F_m coincide?

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Solutions

Theorem [Kharlampovich and Myasnikov (1998-2006), independently Sela (2001-2006)]

$Th(F_n) = Th(F_m), m, n > 1.$

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Examples

Examples of sentences in the theory of F: (Vaught's identity)
$\forall x \forall y \forall z (x^2 y^2 z^2 = 1 \rightarrow$
$([x, y] = 1 \& [x, z] = 1 \& [y, z] = 1))$

(Torsion free) $\forall x (x^n = 1 \rightarrow x = 1)$

(Commutation transitivity)
$\forall x \forall y \forall z ((x \neq 1 \& y \neq 1 \& z \neq 1 \& [x, y] = 1$
$\& [x, z] = 1) \rightarrow [y, z] = 1)$
CT doesn't hold in $F_2 \times F_2$.

Examples

(CSA) $\forall x \forall y([x, x^y] = 1 \to [x, y] = 1)$

$\forall x, y \exists z(xy = yx \to (x = z^2 \vee y = z^2 \vee xy = z^2))$
not true in a free abelian group of rank ≥ 2.
This implies that if a group G is $\forall\exists$ equivalent to F, then it does not have non-cyclic abelian subgroups.

F has Magnus' property, namely, for any n, m the following sentence is true:
$\forall x \forall y(\exists z_1, \ldots, z_{m+n}(x = \prod_{i=1}^{n} z_i^{-1} y^{\pm 1} z_i \wedge y = \prod_{i=n+1}^{m+n} z_i^{-1} x^{\pm 1} z_i) \to \exists z(x = z^{-1} y^{\pm 1} z))$

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Long history of Tarski's type problems in algebra.

Crucial results on fields, groups, boolean algebras, etc.

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Tarski's Type Problems: Complex numbers

Complex numbers $\mathbb{C}$

- $Th(\mathbb{C}) = Th(F)$ iff F is an algebraically closed field.
- $Th(\mathbb{C})$ is decidable.

This led to development of the theory of algebraically closed fields.

Elimination of quantifiers: every formula is logically equivalent (in the theory ACF) to a boolean combination of quantifier-free formulas (something about systems of equations).

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- $Th(\mathbb{R})$ is decidable.

A real closed field = an ordered field where every odd degree polynomial has a root and every element or its negative is a square.

Theory of real closed fields (Artin, Schreier), 17th Hilbert Problem (Artin) (given a psd polynomial $f \in \mathbb{R}[x_l, \ldots, x_k]$, can f be written as a sum of squares of elements in $\mathbb{R}(X)$?)

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Ax-Kochen, Ershov

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- $Th(\mathbb{Q}_p) = Th(F)$ iff F is p-adically closed field.
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Existence of roots of odd degree polynomials in $\mathbb{R} \approx$ Hensel's lemma in $\mathbb{Q}_p$.

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If A_n is a free abelian group of rank n, then
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For non-abelian groups results are sporadic.

Novosibirsk Theorem [Malcev, Ershov, Romanovskii, Noskov]

Let G be a finitely generated solvable group. Then $Th(G)$ is decidable iff G is virtually abelian (finite extension of an abelian group).

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Remeslennikov, Myasnikov, Oger: elementary classification of nilpotent groups (not finished yet).

Typical results:

[Myasnikov, Oger]

Finitely generated non-abelian nilpotent groups G and H are elementarily equivalent iff $G \times \mathbb{Z} \simeq H \times \mathbb{Z}$

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Nothing like that in free groups: no visible logical invariants. Ranks of free non-abelian groups are not definable. Only maximal cyclic subgroups are definable.

Elimination of quantifiers (as we know now): to boolean combinations of $\forall\exists$-formulas!

New methods appeared. These methods allow one to deal with a wide class of groups which are somewhat like free groups: hyperbolic, relatively hyperbolic, acting nicely on Λ-hyperbolic spaces, etc.

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Algebraic sets

G - a group generated by A,
$F(X)$ - free group on $X = \{x_1, x_2, \dots x_n\}$.

A **system of equations** $S(X, A) = 1$ in variables X and coefficients from G (viewed as a subset of $G * F(X)$).

A **solution** of $S(X, A) = 1$ in G is a tuple $(g_1, \dots, g_n) \in G^n$ such that $S(g_1, \dots, g_n) = 1$ in G.

$V_G(S)$, the set of all solutions of $S = 1$ in G, is called an **algebraic set** defined by S.

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Radicals and coordinate groups

The maximal subset $R(S) \subseteq G * F(X)$ with

$$V_G(R(S)) = V_G(S)$$

is the **radical** of $S = 1$ in G.

The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the **coordinate group** of $S = 1$.

Solutions of $S(X) = 1$ in $G \iff$ **G-homomorphisms** $G_{R(S)} \to G$.

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Zariski topology

The following conditions are equivalent:

- G is equationally Noetherian, i.e., every system $S(X) = 1$ over G is equivalent to some finite part of itself.
- the Zariski topology (formed by algebraic sets as a sub-basis of closed sets) over G^n is Noetherian for every n, i.e., every proper descending chain of closed sets in G^n is finite.
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Zariski topology

If the Zariski topology is Noetherian then every algebraic set can be uniquely presented as a finite union of its irreducible components:

$$V = V_1 \cup \ldots V_k.$$

Recall, that a closed subset V is irreducible if it is not a union of two proper closed (in the induced topology) subsets.

The following is an immediate corollary of the decomposition of algebraic sets into their irreducible components.

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Zariski topology

Embedding theorem

Let G be equationally Noetherian. Then for every system of equations $S(X) = 1$ over G there are finitely many irreducible systems $S_1(X) = 1, \ldots, S_m(X) = 1$ (that determine the irreducible components of the algebraic set $V(S)$) such that

$$G_{R(S)} \hookrightarrow G_{R(S_1)} \times \ldots \times G_{R(S_m)}$$

Zariski topology

[R. Bryant (1977), V.Guba (1986)]: Free groups are equationally Noetherian.

Proof Let $H_0 \to H_1 \to \ldots$ be a sequence of epimorphisms between fg groups. Then the sequence

$$Hom(H_0, F) \leftarrow Hom(H_1, F) \leftarrow \ldots$$

eventually stabilizes because
we embed F in $SL_2(\mathbf{Q})$
and the sequence of algebraic varieties

$$Hom(H_0, SL_2(\mathbf{Q})) \leftarrow Hom(H_1, SL_2(\mathbf{Q})) \leftarrow \ldots$$

eventually stabilizes by the Hilbert basis theorem.

Diophantine problem in free groups

Theorem [Makanin, 1982]

There is an algorithm to verify whether a given system of equations has a solution in a free group (free semgroup) or not.

He showed that if there is a solution of an equation $S(X, A) = 1$ in F then there is a "short" solution of length $f(|S|)$ where f is some fixed computable function.

Extremely hard theorem! Now it is viewed as a major achievement in group theory, as well as in computer science.

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Complexity

" Eternity is really long, especially near the end "(Woody Allen)

The original Makanin's algorithm is very inefficient - not even **primitive recursive**.

Plandowski gave a much improved version (for free semigroups): P-**space**.

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Theorem [Kharlampovich, Lysenok, Myasnikov, Touikan (2008)]

The Diophantine problem for quadratic equations in free groups is NP-complete.
[Kh, Vdovina] The length of a minimal solution of a quadratic equation is bounded by $3L^2$, where L is the length of the equation.

Theorem [announced by Lysenok]

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Fragments of elementary theories

The set of all positive sentences which are valid in H is a *positive theory* $Th_+(H)$. The set of all universal sentences is a *universal theory*

Theorem[Mak82] Existential positive theory of F is decidable.

Complexity

Theorem [Makanin]

The existential $Th_{\exists}(F)$ and the universal $Th_{\forall}(F)$ theories of F are decidable.

[Razborov, 85] Description of a solution set of a finite system of equations in a free group.

Complexity

A few years later Edmunds and Commerford and Grigorchuck and Kurchanov described solution sets of arbitrary quadratic equations over free groups. These equations came to group theory from topology and their role in group theory was not altogether clear then. Now they form one of the corner-stones of the theory of equations in groups due to their relations to JSJ-decompositions of groups.

It was known long before that non-abelian free groups have the same existential and universal theories.

Main question was: what are finitely generated groups G with $Th_{\forall}(G) = Th_{\forall}(F)$?

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Groups universally equivalent to F

Unification Theorems

Let G be a finitely generated group and $F \leq G$. Then the following conditions are equivalent:

1) [Remeslennikov] G is universally equivalent to F;
2) G is the coordinate group of an irreducible variety over F.
3) G is discriminated by F (fully residually free), i.e. for any finite subset $M \subseteq G$ there exists a homomorphism $G \longrightarrow F$ injective on M.
4) G is a limit of free groups in Gromov-Grigorchuk metric.
5) G is a Sela's limit group.

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Groups universally equivalent to F

Unification Theorems

Let G be a finitely generated group and $F \leq G$. Then the following conditions are equivalent:

1) [Remeslennikov] G is universally equivalent to F;
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Unification theorems for fully residually free groups

This result shows that the class of fully residually free groups is quite special - it appeared (and was independently studied) in several different areas of group theory.

It turned out that similar results hold for many other groups! (torsion free relatively hyperbolic with abelian parabolics, solvable etc.)

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Groups universally equivalent to F

Lyndon: introduced **free exponential groups** $F^{\mathbb{Z}[t]}$ over polynomials $\mathbb{Z}[t]$ (to describe solution sets of one-variable equations).
He showed also: $F^{\mathbb{Z}[t]}$ *is discriminated by* F*.*

Groups universally equivalent to F

G is an **extension of centralizer** of H if

$$G = \langle H, t \mid [C_H(u), t] = 1 \rangle$$

$\phi : G \to H$ is identical on H and $t^{\phi} = u^n$, for a large n.
Elements in G have canonical form $g_1 t^{m_1} g_2 t^{m_2} \dots t^{m_k} g_{k+1}$, where $g_i \notin C_H(u), i = 1, \dots, k$, and are mapped by ϕ to $g_1 u^{nm_1} g_2 u^{nm_2} \dots u^{nm_k} g_{k+1}$.
If H is discriminated by F, then so is G.

Groups universally equivalent to relatively hyperbolic groups

Theorem[Kharlampovich, Miasnikov, 96] That's it. Every f.g. fully residually free group is a subgroup of a group obtained from a free group as a finite series of extensions of centralizers.

Theorem [Kharlampovich, Miasnikov, 2008] Let Γ be a f.g. torsion free relatively hyperbolic group with abelian parabolic subgroups. Every f.g. fully residually Γ group is a subgroup of a group obtained from Γ as a finite series of extensions of centralizers.

Fully residually free groups

Proposition A frf G has the following properties.

1. G is torsion-free;
2. Each subgroup of G is a fully residually free group;
3. G has the CSA property;
4. Each Abelian subgroup of G is contained in a unique maximal finitely generated Abelian subgroup, in particular, each Abelian subgroup of G is finitely generated;
5. G is finitely presented, and has only finitely many conjugacy classes of its maximal Abelian subgroups.
6. G has solvable word problem;
7. G is linear;
8. Every 2-generated subgroup of G is either free or abelian;
9. If rank $(G)=3$ then either G is free of rank 3, free abelian of

limits of free groups

[Ch. Champetier and V. Guirardel (2004)]
A marked group (G, S) is a group G with a prescribed family of generators $S = (s_1, \ldots, s_n)$.
Two marked groups $(G, (s_1, \ldots, s_n))$ and $(G', (s'_1, \ldots, s'_n))$ are isomorphic as marked groups if the bijection $s_i \longleftrightarrow s'_i$ extends to an isomorphism. For example, $(\langle a \rangle, (1, a))$ and $(\langle a \rangle, (a, 1))$ are not isomorphic as marked groups. Denote by $\mathcal{G}_n$ the set of groups marked by n elements up to isomorphism of marked groups.
One can define a metric on $\mathcal{G}_n$ by setting the distance between two marked groups (G, S) and (G', S') to be e^{-N} if they have exactly the same relations of length at most N (under the bijection $S \longleftrightarrow S'$) (Grigorchuk, Gromov's metric)
Finally, a limit group is a limit (with respect to the metric above) of marked free groups in $\mathcal{G}_n$.

limits of free groups

Example: A free abelian group of rank 2 is a limit of a sequence of cyclic groups with marking

$$(\langle a \rangle, (a, a^n)), \;\; n \to \infty.$$

In the definition of a limit group, F can be replaced by any equationally Noetherian group or algebra.
It is interesting to study limits of free semigroups or algebras.

Limit groups by Sela

Bestvina, Feighn's reformulation H fg, a sequence $\{\phi_i\}$ in $Hom(H, F)$ is stable if, for all $h \in H$, h^{ϕ_i} is eventually always 1 or eventually never 1.
$\underrightarrow{Ker\ \phi_i}$ is:

$$\{h \in H | h^{\phi_i} = 1 \text{ for almost all } i\}$$

G is a **limit group** if there is a fg H and a stable sequence $\{\phi_i\}$ such that:

$$G \cong H/\underrightarrow{Ker\ \phi_i}.$$

Krull dimension

Theorem [announced by Louder] There exists a function $g(n)$ such that the length of every proper descending chain of closed sets in F^n is bounded by $g(n)$.

Homogeneity of F

Let M be a model, P a subset of M and $\bar{a}$ a tuple from M. The type of $\bar{a}$ over P, denoted $tp^M(\bar{a}|P)$ is the set of all formulas $\phi(\bar{x})$ with coefficients from P such that M satisfies $\phi(\bar{a})$.

Theorem (C. Perin & R. Sklinos, A. Ould Houcine)
Any nonabelian free group F_n of finite rank is homogeneous; that is for any tuples $\bar{a}, \bar{b}$, of F_n, having the same complete n-type, there exists an automorphism of F_n which sends $\bar{a}$ to $\bar{b}$.

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Homogeneity of F

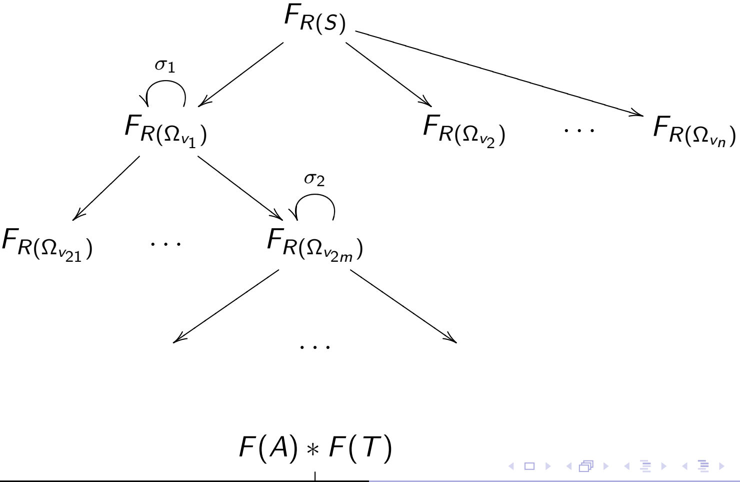
If g is an element of F_n such that F_n is freely indecomposable with respect to g, then the type of g is not realized by any element of F_k for $k < n$. To see this, just note that if there was such an element h in F_k, we would have $tp^{F_k}(h) = tp^{F_n}(h)$ (since F_k elementary in F_n) and thus $tp^{F_n}(h) = tp^{F_n}(g)$. By homogeneity, there is an automorphism of F_n sending g to h. But h lies in F_k which is a proper free factor, this contradicts indecomposability of F_n with respect to g.

Description of solutions

Theorem [Razborov]

Given a finite system of equations $S(X) = 1$ in $F(A)$ one can effectively construct a finite Solution Diagram that describes all solutions of $S(X) = 1$ in F.

Solution diagrams



Elimination Process

Kharlampovich - Myasnikov (1998): introduced a modification of Razborov's process, called an Elimination Process (EP) to describe solutions of systems of equations in free groups in terms of **triangular quadratic systems** of equations. This solves (in the strongest possible form) an open problem posed by Razborov.

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Effectiveness of Grushko's and JSJ decompositions

Theorem [KM]

There is an algorithm which for every finitely generated fully residually free group finds its Grushko's decomposition (by giving finite generating sets of the factors).

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There exists an algorithm to obtain a cyclic [abelian] JSJ decomposition of a freely indecomposable fully residually free group. The algorithm constructs a presentation of this group as the fundamental group of a JSJ graph of groups.

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Isomorphism problem

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The isomorphism problem is decidable in the class of all finitely generated fully residually free groups.

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Quadratic equations

Up to an automorphism every quadratic equation $S = 1$ over a group G takes one of the following forms:

$$\prod_{i=1}^{n}[x_i, y_i]\prod_{i=1}^{m} z_i^{-1}c_i z_i d = 1, \quad n, m \geq 0, m+n \geq 1;$$

$$\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1}c_i z_i d = 1, \quad n, m \geq 0, n+m \geq 1.$$

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Words of the type $[x, y]$, x^2, $z^{-1}cz$, are called **atoms**,

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Regular quadratic equations

A solution $\phi : G_{R(S)} \to G$ of $S = 1$ in G is called **commutative** if

$$[r_i^{\phi}, r_{i+1}^{\phi}] = 1$$

for all consecutive atoms r_i, r_{i+1} of $S = 1$.

$S = 1$ is **regular** if either it is an equation of the type $[x, y] = d$ ($d \neq 1$), or the equation $[x_1, y_1][x_2, y_2] = 1$, or $r(S) \geq 2$ and $S(X) = 1$ has a non-commutative solution and it is not an equation of the type $c_1^{z_1} c_2^{z_2} = c_1 c_2$, $x^2 c^z = a^2 c$, $x_1^2 x_2^2 = a_1^2 a_2^2$.

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A triangular quasi-quadratic (TQ) system has the following form

$$S_1(X_1, X_2, \ldots, X_n, A) = 1,$$
$$S_2(X_2, \ldots, X_n, A) = 1,$$
$$\ldots$$
$$S_n(X_n, A) = 1$$

where S_i is either quadratic in variables X_i, or corresponds to an extension of a centralizer, or to an abelian extension, or empty.
A TQ system is **non-degenerate (NTQ)** if for every i the equation $S_i(X_i, \ldots, X_n, A) = 1$ has a solution in the coordinate group $F_{R(S_{i+1},\ldots,S_n)}$, where $X_{i+1}, \ldots, X_n, A$ are viewed as constants.

Description of solutions

Theorem [KM, 98]

Given an arbitrary system $S(X, A) = 1$ EP starts on $S(X, A)$ and outputs finitely many NTQ systems

$$U_1(Y) = 1, \ldots, U_m(Y) = 1$$

such that

$$V_F(S) = P_1(V(U_1)) \cup \ldots \cup P_m((U_m))$$

for some word mappings $P_1, \ldots, P_m$.

Corollary

Up to the rational equivalence algebraic sets over F are finite unions of sets defined by NTQ systems.

Elementary free groups

Theorem [KM and S, published 2006]

A finitely generated group which is $\forall\exists$-equivalent to a free non-abelian group F is isomorphic to the coordinate group of a regular NTQ system over F.

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Discrete non-Archimedean actions

Morgan and Shalen (1985) defined Λ-trees:
A Λ-tree is a metric space (X, p) (where $p : X \times X \to \Lambda$) which satisfies the following properties:

1) (X, p) is geodesic,
2) if two segments of (X, p) intersect in a single point, which is an endpoint of both, then their union is a segment,
3) the intersection of two segments with a common endpoint is also a segment.

Theorem [Kharlampovich, Myasnikov]

Every finitely generated fully residually free group acts freely on some $\mathbb{Z}^n$-tree for a suitable n.

Open Problem [Rips, Bass]

Finitely generated $\mathbb{R}$-free groups

Rips' Theorem [Rips, 1991 - not published]

A f.g. group acts freely on $\mathbb{R}$-tree if and only if it is a free product of surface groups (except for the non-orientable surfaces of genus 1,2, 3) and free abelian groups of finite rank.

The Fundamental Problem

The following is a principal step in the Alperin-Bass' program:

Open Problem [Rips, Bass]

Describe finitely generated groups acting freely on Λ-trees.

Here "describe" means "describe in the standard group-theoretic terms".

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The Main Conjecture.

Conjecture

Every finitely generated Λ-free group is $\mathbb{Z}^n$-free.

Finitely presented complete Λ-free groups.

Theorem [Kharlampovich, Myasnikov, Serbin]

If G is f.p. complete (all branch points are in the same orbit) Λ-free group, then G has an index two subgroup that can be represented as a union of a finite series of groups

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian and length-isomorphic.

Theorem [Kharlampovich, Myasnikov, Serbin]

Any f.p. Λ-free group G is $\mathbb{R}^k$-free for an appropriate k, where $\mathbb{R}^k$ is ordered lexicographically .