Cornell Topology Festival Panel Discussion

May 2011

On May 8th, 2011, the speakers from the Cornell Topology Festival gave a panel discussion on recent developments (other than their own work) and interesting questions in topology and related fields. The speakers were asked to limit their presentations to 5 minutes. Notes were taken and complied by Cornell graduate students. Any errors in what follows are almost certainly due to us, and not the speakers.

Matthew Foreman (University of California at Irvine)

Foreman presented a summary of results about entropy for measure-preserving actions. This quantity measures the disorder for $\mathbb{Z}$-actions, however he illustrated with an example that entropy fails to capture the notion of disorder for actions of arbitrary groups.

Let (K, ν) be a finite or countable measure space and let G be a countable group, K^G with the product measure forms a measure space and G acts on it by letting $(g \cdot f)(h) = f(g^{-1}h)$ for $g \in G$ and $f \in K^G$. This action is called a Bernoulli shift.

Kolmogorov and Sinai defined entropy for measure-preserving $\mathbb{Z}$-actions. For Bernoulli shifts, the entropy is given by $h(K^G) = -\sum_{k \in K} \nu(\{k\}) \log \nu(\{k\})$. Ornstein proved that this quantity is a complete invariant for measure-preserving $\mathbb{Z}$-actions, that is, he proved that two actions are isomorphic if and only if they have the same entropy. He later extended the definition of entropy for measure-preserving actions of amenable groups, which was further extended by Bowen to measure-preserving actions of sofic groups. However, outside the class of $\mathbb{Z}$-actions, entropy fails to measure the amount of disorder as the following example shows.

Consider the free group F_2 on two generators a, b. It can be checked that $\phi : (\mathbb{Z}_2)^{F_2} \to (\mathbb{Z}_2 \times \mathbb{Z}_2)^{F_2}$ defined by

$$\phi(x)(g) = (x(g) + x(ga), x(g) + x(gb))$$

is a factor map. However, the entropy of $(\mathbb{Z}_2)^{F_2}$ is greater than the entropy of $(\mathbb{Z}_2 \times \mathbb{Z}_2)^{F_2}$ contrary to what we would expect.

Jason Manning (SUNY at Buffalo)

Let $\underline{w} = \{w_1, \ldots, w_k\}$ be a finite collection of conjugacy classes in the free group F_n. It can be realized by an embedding ϕ of a disjoint union of k circles into a genus n 3-dimensional handlebody W. We say that $\underline{w}$ is *geometric* if φ is homotopic to an embedding into ∂W. More generally, $\underline{w}$ is *virtually geometric* if there is a finite cover $\widetilde{W} \xrightarrow{\pi} W$ so that the inclusion of the preimage $\widetilde{S}$ of the embedded circles in $\widetilde{W}$ is homotopic to an embedding of $\widetilde{S}$ in $\partial\widetilde{W}$.

$$\begin{array}{ccc} \widetilde{S} & \hookrightarrow & \widetilde{W} \\ \downarrow & & \downarrow \pi \\ S & \xhookrightarrow{\varphi} & W \end{array}$$

Gordon-Wilton showed some words are virtually geometric but not geometric, for instance the Baumslag-Solitar words $ab^pa^{-1}b^{-q}$. Jason Manning showed there are words that are not even virtually geometric.

A conjugacy class fixes a pair of points in the boundary of a free group. Otal studied the space formed from the boundary of the free group F_n by smashing togething these pairs corresponding to $\underline{w}$. Chris Cashen proved that the space is planar iff $\underline{w}$ is virtually geometric.

Olga Kharlampovic (McGill University)

Question. *Prove that the class of limit groups is not rigid. Does there exist a group G quasi-isometric to a limit group, but not virtually a limit group?*

Two simple questions,

1) Consider $G_1 = \langle F_1, t \mid [w_1, t] = 1\rangle, G_2 = \langle F_2, t \mid [w_2, t] = 1\rangle$ such that w_1, w_2 are not proper powers. Can one construct (G_1, G_2), such that G_1 is quasi-isometric to G_2 and w_1 is not an automorphic image of w_2?

2) If instead of extentions of centralizers we take a double, can one construct different words such that the groups are quasi-isometric?

The work of Cashen produces some examples of the above.

Simon Thomas (Rutgers University)

Simon Thomas discussed a recent result of Anna Erschler on finitely generated amenable groups.

Let G be finitely generated amenable group and let $\Gamma = \mathrm{Cay}(G, S)$ be its Cayley graph.

The Cayley graph is amenable, i.e. we can find the Følner function $F_{G,S}(k) = \min |V|$ such that $\frac{|\partial V|}{|V|} < \frac{1}{k}$, where V is the set of vertices. The growth rate in the Følner function is quasi-isometry invariant, so we can just refer to F_G. This invariant has the following monotonicity property: If H is any finitely generated group that embeds in G then $F_H \preceq F_G$.

Theorem (Erschler)**.** *For every $f : \mathbb{N} \to \mathbb{N}$, there is a finitely generated amenable group G such that $f \preceq F_G$.*

Corollary. *There does not exist a universal finitely generated amenable group up to quasi-isometry.*

Question. *In the class of finitely generated groups (not necessarily amenable), is there a quasi-isometry invariant which satisfies the corresponding monotonicity property?*

Justin Moore, another speaker at the conference, also added the following related question by Gromov:

Question. *Is there an amenable finitely presented group with non-primitive recursive Følner function? Is Thompson's group such a group?*

Justin Moore (Cornell University)

Question. *Is there an amenable finitely presented group with a non primitive recursive Følner function?*

Conjecture (Von Neumann/Day)**.** *If G is non-amenable, G contains F_2.*

The above was shown to be false by a construction due to Olshanskii.

Theorem (Gaborian/Lyons)**.** *If G is non-amenable, finitely generated, then there is an ergodic measure preserving action of G on a probability space (X, μ) by measure preserving transformations, and another ergodic action of F_2 on (X, μ) such that the orbits of the action of G are unions of the orbits of the action of F_2.*

Darren Long (University of California at Santa Barbara)

Let F be a closed orientable surface and $\theta : F \to F$ be a pseudo-Anosov automorphism, with dilation $\lambda(F)$. This is an algebraic unit. Dilation satisfies that $deg(f) \leq 6 \cdot g(F) - 6$, where $g(F)$ is the genus of F. An entropy is asociated to θ, given by $\log(\lambda)$.

The dilation is found by looking at particular branched coverings of F.

Question. *What are the constraints on λ?*

Bestvina-Handel's train-track maps are an invariant for (θ, λ). If λ is a dilation of a psuedo-Anosov map, then it must be the spectral radius of a symplectic, integral matrix representation of such a map.

Question. *If λ is pairing Frobenius, the spectral radius of a matrix, symplectic and integral, is it the dilation of a psuedo-Anasov map?*

This is being investigated by Robert Ackermann.

Yair Minsky (Yale University)

This is a short introduction of the work by Kim-Koberda about RAAGs.

A RAAG (Right-angled Artin group) $A(\Gamma)$ associated to a finite graph Γ is a group such that the set of vertices of Γ form a generating set and two generators commute if there is an edge between two corresponding vertices in Γ.

The following two examples show two special cases.

- if Γ_1 is the complete graph with n vertices, then $A(\Gamma_1)$ is $\mathbb{Z}^n$.
- if Γ_2 consists of n vertices with no edge, then $A(\Gamma_2)$ is F_n, the free group on n generators.

One natural question is the following:
For two graphs Γ_1, Γ_2, when does $A(\Gamma_1)$ embed as a subgroup of $A(\Gamma_2)$?

Given a finite graph Γ, we define Γ^e as the following:

- vertices are conjugates of vertices of Γ
- put an edge between any two commuting vertices

In particular, in the Abelian group case, you get $\Gamma = \Gamma^e$.

Theorem. *If Γ_1 embeds in Γ_2^e, then $A(\Gamma_1)$ is a subgroup of $A(\Gamma_2)$.*

Conjecture. *The converse also holds.*

Idea for proof: Realize $A(\Gamma)$ as generated by Dehn twists of curves on some surface. The adjoint graph of the curve graph gives you Γ^e.

Slawek Solecki (University of Illinois at Urbana-Champaign)

Slawek Solecki presented a result by Su Gao and Steve Jackson on hyperfinite equivalence relations. An equivalence relation E on a standard Borel space is said to be **hyperfinite** if E is the increasing union of countably many Borel equivalence relations E_n, where all E_n-equivalence classes are finite. It is known that any hyperfinite equivalence relation is the orbit equivalence relation of a Borel action of $\mathbb{Z}$, and any Borel $\mathbb{Z}$-action gives rise to a hyperfinite equivalence relation. Jackson and Gao answered the question "For which groups G does every Borel G-action give rise to *only* hyperfinite orbit equivalence relations?"

Theorem (Gao, Jackson)**.** *Every orbit equivalence relation of a Borel action of a countable abelian group on a standard Borel space is hyperfinite.*

As an application of this theorem, Solecki mentioned the equivalence relation F_0 defined on $\mathbb{R}_{\geq 0}$ by $x\ F_0\ y$ if and only if $\frac{x}{y} \in \mathbb{Q}$. Gao and Jackson's theorem implies that F_0 is hyperfinite.

Alexandra Pettet (Oxford University)

The rank n free group F_n is analogous to the fundamental group $\pi_1 S_g$ of a genus g closed orientable surface, with $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$ playing a role similar to the mapping class group Mod_g.

Magnus studied the *Identity on Abelianization* (IA) subgroup of $\mathrm{Aut}(F_n)$, the kernel

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \to 1,$$

and proved that it is finitely generated (a minimal generating set requires a cubic in n generators.) Johnson later proved the analogous *Torelli* subgroup T_g of the mapping class group, defined as

$$1 \to \mathrm{T}_g \to \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1,$$

is finitely generated. But the smallest known generating set is exponential in g, while it should be linear.

Recall the lower central series for a group G:

$$G = G_1 \geq G_2 \geq \cdots \geq G_{k+1} = [G, G_k] \geq \cdots$$

Johnson studied a filtration of the Torelli group; the analogous *Johnson filtration* of IA_n is $J_n^k := \ker(\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/(F_n)_{k+1})$. For example, $J_n^1 = \mathrm{IA}_n$ and $J_n^2 = [\mathrm{IA}_n, \mathrm{IA}_n]$. A natural open question is:

Question. *Is J_n^k finitely generated?*

Some progress:

Theorem (Papadima-Saciu 2010)**.** $\dim_{\mathbb{Q}} H^1(J_n^2, \mathbb{Q})$ *is (surprisingly) finite for $n \geq 5$.*

Note: there must be a requirement on n: $\mathrm{IA}_2 = F_2$, so $[\mathrm{IA}_2, \mathrm{IA}_2]$ is an infinite rank free group.