The Complexity of the Quasi-Isometry Relation for Finitely Generated Groups

Simon Thomas

Rutgers University

8th May 2011

The Main Theorems ...

Theorem

There does not exist an Borel choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.

Theorem

The quasi-isometry relation $\approx_{QI}$ on the space $\mathcal{G}_{fg}$ of finitely generated groups is not smooth.

Conjecture

- *The quasi-isometry relation $\approx_{QI}$ on $\mathcal{G}_{fg}$ is a universal $\mathbf{K}_\sigma$ equivalence relation.*
- *In particular, the problem of classifying f.g. groups up to quasi-isometry is strictly harder than that of classifying them up to isomorphism.*

A slight digression: The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann 1949)

Every countable group G can be embedded into a 2-generator group.

Sketch Proof.

- Let $(\, g_n \mid n \in \mathbb{N}\,)$ be a sequence of generators with $g_0 = 1$.
- Let $\mathbb{F}$ be the free group on $\{\, a, b \,\}$ and let $G * \mathbb{F}$ be the free product.
- Then $\{\, b^{-n}ab^n \mid n \in \mathbb{N} \,\}$ and $\{\, g_n a^{-n} b a^n \mid n \in \mathbb{N} \,\}$ freely generate free subgroups of $G * \mathbb{F}$.
- Hence we can construct the *HNN* extension

$$G \hookrightarrow K_G = \langle\, G * \mathbb{F}, t \mid t^{-1} b^{-n} a b^n t = g_n a^{-n} b a^n \,\rangle$$

- Since $g_n \in \langle a, b, t \rangle$ and $t^{-1} a t = b$, it follows that $K_G = \langle a, t \rangle$.

$\square$

A natural question

Observation

It is reasonably clear that the isomorphism type of the 2-generator group K_G usually depends upon both the generating set of G and the particular enumeration that is used.

Question

Does there exist a more uniform construction with the property that the isomorphism type of K_G only depends upon the isomorphism type of G?

The answer ...

Notation

$\mathcal{G}$ and $\mathcal{G}_{fg}$ denotes the spaces of countable groups and f.g. groups.

Theorem

There does not exist a Borel map $G \mapsto K_G$ from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that for all $G, H \in \mathcal{G}$,

- $G \hookrightarrow K_G$; *and*
- *if* $G \cong H$, *then* $K_G \cong K_H$.

But Greg Cherlin wasn't satisfied ...

Theorem

- *Suppose that $G \mapsto K_G$ is* any *Borel map from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$.*
- *Then there exists an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are incomparable with respect with respect to embeddability.*

Sketch proof.

Collapse the continuum $\mathbb{R}$ to a countable set and then apply the Shoenfield Absoluteness Theorem. □

Back to the actual talk ...

Definition

Let G be a f.g. group and let $S \subseteq G \smallsetminus \{1_G\}$ be a finite generating set. Then the Cayley graph $\mathrm{Cay}(G, S)$ *is the graph with vertex set G and edge set*

$$E = \{\{x, y\} \mid y = xs \text{ for some } s \in S \cup S^{-1}\}.$$

The corresponding word metric *is denoted by d_S.*

Main Theorem

There does not exist an Borel *choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.*

A reason to believe ...

Question

Why is the Main Theorem "obviously true"?

Answer

Because the isomorphism problem for f.g. groups is much harder than that for Cayley graphs.

Borel reductions

Definition

Let E, F be equivalence relations on the Polish spaces X, Y.

- $E \leq_B F$ *if there exists a Borel map* $\varphi : X \to Y$ *such that*

  $$x \, E \, y \iff \varphi(x) \, F \, \varphi(y).$$

  In this case, φ *is called a* ***Borel reduction*** *from* E *to* F.
- $E \sim_B F$ *if both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *if both* $E \leq_B F$ *and* $E \nsim_B F$.

Smooth vs Nonsmooth

Definition

The equivalence relation E on the Polish space X is smooth if there exists a Borel map $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ such that

$$x \, E \, y \iff \varphi(x) = \varphi(y).$$

Example

The classification problem for countable divisible abelian groups is smooth.

Nonexample

The classification problem for f.g. groups is not smooth.

Vertex transitive graphs of finite valency

Definition

Let $\mathcal{C}$ be the Polish space of graphs Γ with underlying set $\mathbb{N}$ which satisfy the following conditions:

- *Each vertex $v \in \Gamma$ has finite degree.*
- $\mathrm{Aut}(\Gamma)$ *acts transitively on* Γ.

Observation

$\mathcal{C}$ includes the Cayley graphs of f.g. groups.

Theorem (Folklore)

The isomorphism relation on $\mathcal{C}$ is smooth.

Borel homomorphisms

Definition

The Borel map $\varphi : X \to Y$ is a ***homomorphism*** *from E to F if*

$$x \, E \, y \Longrightarrow \varphi(x) \, F \, \varphi(y).$$

Main Theorem

*If $\varphi : \langle \mathcal{G}_{fg}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$ is **any** Borel homomorphism, then there exist groups G, $H \in \mathcal{G}_{fg}$ such that:*

- $\varphi(G) \cong \varphi(H)$.
- *G and H don't have isomorphic Cayley graphs.*

Heuristic Reason

Since $\cong_{\mathcal{G}_{fg}}$ is **much more complex** than $\cong_{\mathcal{C}}$, the map φ has a "large kernel" and so "too many" groups G will be mapped to a fixed graph Γ.

Borel homomorphisms

Definition

The Borel map $\varphi : X \to Y$ is a homomorphism from E to F if

$$x \, E \, y \implies \varphi(x) \, F \, \varphi(y).$$

Main Theorem

If $\varphi : \langle \mathcal{G}_{fg}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$ is any Borel homomorphism, then there exist groups G, $H \in \mathcal{G}_{fg}$ such that:

- $\varphi(G) \cong \varphi(H)$.
- *G and H don't have isomorphic Cayley graphs.*

Question

But how can we be sure that two f.g. groups don't have isomorphic Cayley graphs with respect to some finite generating sets?

Almost there ...

Main Theorem

If $\varphi : \langle \mathcal{G}_{fg}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$ is any Borel homomorphism, then there exist groups $G, H \in \mathcal{G}_{fg}$ such that:

- $\varphi(G) \cong \varphi(H)$.
- *G and H are not quasi-isometric.*

Remark

Of course, in order to prove this, we must actually show that there are "many such pairs".

A little ergodic theory

Definition

A measure preserving action of a group $\mathbb{G}$ on a probability space (X, ν) is ergodic *if whenever $Y \subseteq X$ is a $\mathbb{G}$-invariant Borel subset, then $\nu(Y) = 0, 1$.*

Remark

Equivalently, if $\psi : X \to \mathbb{N}^{\mathbb{N}}$ is a $\mathbb{G}$-invariant Borel function, then there exists a Borel subset $M \subseteq X$ with $\nu(M) = 1$ such that $\psi \restriction M$ is a constant function.

Example

Let μ be the usual product probability measure on $2^{\mathbb{Z}}$ and consider the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}} = \mathcal{P}(\mathbb{Z})$. Then μ is $\mathbb{Z}$-invariant and $\mathbb{Z}$ acts ergodically on $(2^{\mathbb{Z}}, \mu)$.

Some easy consequences

Definition

Let $E_{\mathbb{Z}}$ be the orbit equivalence relation for the action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$.

Remark

$E_{\mathbb{Z}}$ is Borel bireducible with E_0.

Proposition

If $\psi : \langle\, 2^{\mathbb{Z}}, E_{\mathbb{Z}}\,\rangle \to \langle\, \mathbb{N}^{\mathbb{N}}, =\,\rangle$ is a Borel homomorphism, then there exists a Borel subset $M \subseteq 2^{\mathbb{Z}}$ with $\mu(M) = 1$ such that $\psi \restriction M$ is a constant function.

Corollary

If $\psi : \langle\, 2^{\mathbb{Z}}, E_{\mathbb{Z}}\,\rangle \to \langle\, \mathcal{C}, \cong\,\rangle$ is a Borel homomorphism, then there exists a Borel subset $M \subseteq 2^{\mathbb{Z}}$ with $\mu(M) = 1$ such that ψ maps M into a single $\cong$-class.

The heart of the matter

Main Lemma

There exists a Borel homomorphism

$$\theta : \langle\, 2^{\mathbb{Z}}, E_{\mathbb{Z}} \,\rangle \to \langle\, \mathcal{G}_{fg}, \cong \,\rangle$$
$$T \mapsto G_T$$

such that the set

$$\{ \langle S, T \rangle \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \mid G_S, G_T \text{ are not quasi-isometric} \,\}$$

has $(\mu \times \mu)$*-measure 1.*

Proof of Main Theorem

- Suppose that $\varphi : \langle \mathcal{G}_{fg}, \cong \rangle \to \langle \mathcal{C}, \cong \rangle$ is a Borel homomorphism.
- Consider the composite Borel homomorphism

$$\psi : \langle 2^{\mathbb{Z}}, E_{\mathbb{Z}} \rangle \xrightarrow{\theta} \langle \mathcal{G}_{fg}, \cong \rangle \xrightarrow{\varphi} \langle \mathcal{C}, \cong \rangle.$$

- Then there exists a Borel subset $M \subseteq 2^{\mathbb{Z}}$ with $\mu(M) = 1$ such that ψ maps M into a single $\cong$-class of graphs.
- Since the set

$$\{\langle S, T \rangle \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \mid G_S, G_T \text{ are not quasi-isometric } \}$$

has $(\mu \times \mu)$-measure 1, there exist S, $T \in M$ such that G_S, G_T are not quasi-isometric.

The quasi-isometry relation isn't smooth

- Suppose that $\varphi : \langle\, \mathcal{G}_{fg}, \approx_{QI} \,\rangle \to \langle\, \mathbb{N}^{\mathbb{N}}, = \,\rangle$ is a Borel reduction.
- Consider the composite Borel homomorphism

$$\psi : \langle\, 2^{\mathbb{Z}}, E_{\mathbb{Z}} \,\rangle \xrightarrow{\theta} \langle\, \mathcal{G}_{fg}, \approx_{QI} \,\rangle \xrightarrow{\varphi} \langle\, \mathbb{N}^{\mathbb{N}}, = \,\rangle.$$

- Then there exists a Borel subset $M \subseteq 2^{\mathbb{Z}}$ with $\mu(M) = 1$ such that ψ maps M into a single $r \in \mathbb{N}^{\mathbb{N}}$.
- Since the set

$$\{\langle S, T\rangle \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \mid G_S, G_T \text{ are not quasi-isometric } \}$$

has $(\mu \times \mu)$-measure 1, there exist $S, T \in M$ such that G_S, G_T are not quasi-isometric, which is a contradiction.

A slight variant of Champetier's construction

Definition

Let $\mathbb{F}_3$ be the free group on $\{ a, b, c \}$ and let $g \in \mathrm{Aut}(\mathbb{F}_3)$ be the automorphism defined by:

$$g(a) = ab \qquad g(b) = ab^2 \qquad g(c) = c$$

Lemma

If $w = (ac)^{14}$, then the presentation $\langle a, b, c \mid g^n(w) \rangle_{n \in \mathbb{Z}}$ satisfies the $C'(1/6)$ cancellation property.

Definition

Let $\theta : 2^{\mathbb{Z}} \to \mathcal{G}_{fg}$ be the Borel map defined by

$$S \overset{\theta}{\mapsto} G_S = \langle a, b, c \mid \mathcal{R}_S \rangle, \quad \text{where } \mathcal{R}_S = \{ g^s(w) \mid s \in S \}.$$

θ is a Borel homomorphism

Lemma

If $S \; E_{\mathbb{Z}} \; T$, then $G_S \cong G_T$.

Proof.

- Suppose that $T = n + S$ and consider $g^n \in \mathrm{Aut}(\mathbb{F}_3)$.
- Clearly $g^n(\{ g^s(w) \mid s \in S \}) = \{ g^t(w) \mid t \in T \}$.
- Hence g^n induces an isomorphism from $G_S = \langle a, b, c \mid \mathcal{R}_S \rangle$ onto $G_T = \langle a, b, c \mid \mathcal{R}_T \rangle$.

□

Combining ideas of Champetier and Bowditch

Lemma

The "taut loops" in the Cayley graph of $G_S = \langle\, a, b, c \mid \mathcal{R}_S \,\rangle$ *correspond precisely to the relators* $\mathcal{R}_S = \{\, g^s(w) \mid s \in S \,\}$.

Lemma

If G_S, G_T *are quasi-isometric, then*

$$\{\, \text{length}(g^s(w)) \mid s \in S \,\} \equiv \{\, \text{length}(g^t(w)) \mid t \in T \,\}$$

Definition

If $D, E \subseteq \mathbb{N}^+$, then $D \equiv E$, if there exists $k \geq 1$ such that:

- For every $d \in D$, there exists $e \in E$ such that $d/k \leq e \leq kd$.
- For every $e \in E$, there exists $d \in D$ such that $e/k \leq d \leq ke$.

Combining ideas of Champetier and Bowditch

Corollary

The set $\{\langle S, T\rangle \in 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \mid G_S, G_T$ *are not quasi-isometric* $\}$ *has* $(\mu \times \mu)$*-measure 1.*

- It is easily seen that length$(g^{\ell}(w)) \approx 2^{|\ell|}$.
- For $\mu-$a.e. $S \in 2^{\mathbb{Z}}$, for each $k \geq 1$, there exist infinitely many pairwise disjoint "double intervals" $D_n^k(S) \subseteq \mathbb{Z}$ with $|D_n^k(S)| = 4k + 2$ such that $D_n^k(S) \cap S = \emptyset$.
- Fix some such $S \in 2^{\mathbb{Z}}$.
- For $\mu-$a.e. $T \in 2^{\mathbb{Z}}$, for each $k \geq 1$, there exist infinitely many n such that middle$^+(D_n^k(S)) \in T$.
- Clearly G_S, G_T are not quasi-isometric.

The complexity of the quasi-isometry relation

Theorem

The quasi-isometry relation $\approx_{QI}$ on the space $\mathcal{G}_{fg}$ of f.g. groups is not smooth.

Conjecture

- *The problem of classifying f.g. groups up to quasi-isometry is strictly harder than that of classifying them up to isomorphism.*
- *In particular, $E_{\mathbb{Z}} <_B \approx_{QI}$.*

Some more ergodic theory

Definition

Let $E_{\mathbb{F}_2}$ be the orbit equivalence relation for the shift action of $\mathbb{F}_2$ on $(2^{\mathbb{F}_2}, \nu)$, where ν is the usual product probability measure.

Proposition (Jones-Schmidt)

If $\psi : \langle 2^{\mathbb{F}_2}, E_{\mathbb{F}_2} \rangle \to \langle 2^{\mathbb{Z}}, E_{\mathbb{Z}} \rangle$ is a Borel homomorphism, then there exists a Borel subset $M \subseteq 2^{\mathbb{F}_2}$ with $\nu(M) = 1$ such that ψ maps M into a single $E_{\mathbb{Z}}$-class.

What we need ...

Conjecture

There exists a Borel homomorphism

$$\theta : \langle\, 2^{\mathbb{F}_2}, E_{\mathbb{F}_2} \,\rangle \rightarrow \langle\, \mathcal{G}_{fg}, \cong \,\rangle$$
$$S \mapsto G_S$$

such that the set

$$\{\langle S, T \,\rangle \in 2^{\mathbb{F}_2} \times 2^{\mathbb{F}_2} \mid G_S, G_T \text{ are not quasi-isometric} \,\}$$

has $(\nu \times \nu)$*-measure 1.*

More precisely ...

Definition

Let $\mathbb{F}_3$ be the free group on $\{ a, b, c \}$ and let $g, h \in \mathrm{Aut}(\mathbb{F}_3)$ be the automorphisms defined by:

$$g(a) = ab \qquad h(a) = a^2ba^3b$$
$$g(b) = ab^2 \qquad h(b) = a^2b$$
$$g(c) = c \qquad h(c) = c$$

Lemma (Magnus-Neumann)

$F = \langle g, h \rangle$ is a free subgroup of $\mathrm{Aut}(\mathbb{F}_3)$.

Lemma

If $w = (ac)^{14}$, then the presentation $\langle a, b, c \mid \varphi(w) \rangle_{\varphi \in F}$ satisfies the $C'(1/6)$ cancellation property.

More precisely ...

- Consider the Borel homomorphism $2^F \to \mathcal{G}_{fg}$ defined by

$$S \mapsto G_S = \langle\, a, b, c \mid \mathcal{R}_S \,\rangle, \qquad \text{where } \mathcal{R}_S = \{\, \varphi(w) \mid \varphi \in S \,\}.$$

Conjecture

The set

$$Z = \{\langle S, T\rangle \in 2^F \times 2^F \mid G_S, G_T \text{ are not quasi-isometric}\,\}$$

has $(\nu \times \nu)$-measure 1.

Some Universality Questions

Question (Universal group with respect to quasi-isometry?)

Does there exist a f.g. group G such that for every f.g. group K, there exists a f.g. group H such that $K \hookrightarrow H$ and $H \approx_{QI} G$?

Question (Universal Cayley graph?)

Does there exist a fixed graph Γ such that for every f.g. group K, there exists a f.g. group H with generating set S such that $K \hookrightarrow H$ and $\mathrm{Cay}(H, S) \cong \Gamma$?

Question (Universal locally compact group?)

Does there exist a locally compact second countable group $\mathbb{G}$ such that every countable group K embeds into a cocompact lattice of $\mathbb{G}$?