# A Descriptive View of Geometric Group Theory

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# Cayley graphs of finitely generated groups

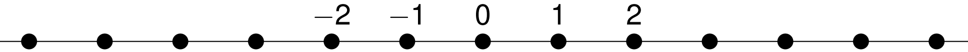
## Definition

*Let $G$ be a f.g. group and let $S \subseteq G \smallsetminus \{1_G\}$ be a finite generating set. Then the* *Cayley graph* $\mathrm{Cay}(G,S)$ *is the graph with vertex set $G$ and edge set*

$$E = \{\{x,y\} \mid y = xs \text{ for some } s \in S \cup S^{-1}\}.$$

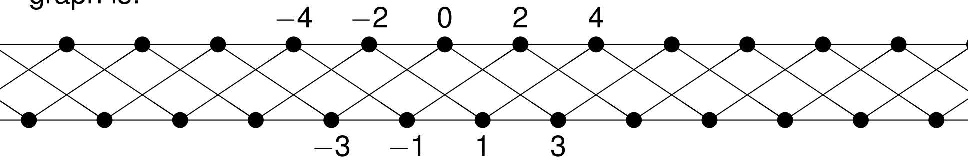
*The corresponding* *word metric* *is denoted by $d_S$.*

For example, when $G = \mathbb{Z}$ and $S = \{1\}$, then the corresponding Cayley graph is:



# But which Cayley graph?

However, when $G = \mathbb{Z}$ and $S = \{2, 3\}$, then the corresponding Cayley graph is:

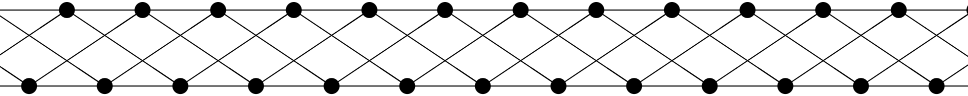



**Theorem (S.T.)**

*There does not exist an explicit choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.*

# The basic idea of geometric group theory

Although the Cayley graphs of a f.g. group $G$ with respect to different generating sets $S$ are usually nonisomorphic, they always have the same large scale geometry.



# The quasi-isometry relation

## Definition (Gromov)

*Let $G$, $H$ be f.g. groups with word metrics $d_S$, $d_T$ respectively. Then $G$, $H$ are said to be quasi-isometric, written $G \approx_{QI} H$, if there exist*

- *constants $\lambda \geq 1$ and $C \geq 0$, and*
- *a map $\varphi : G \to H$*

*such that for all $x, y \in G$,*

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

*and for all $z \in H$,*

$$d_T(z, \varphi[G]) \leq C.$$

# When $C = 0$

## Definition (Gromov)

*Let G, H be f.g. groups with word metrics $d_S$, $d_T$ respectively. Then G, H are said to be Lipschitz equivalent if there exist*

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# As expected ...

## Observation

*If $S$, $S'$ are finite generating sets for $G$, then*

$$id : \langle G, d_S \rangle \longrightarrow \langle G, d_{S'} \rangle$$

*is a quasi-isometry.*

Thus while it doesn't make sense to talk about the isomorphism type of "the Cayley graph of $G$", it does make sense to talk about the quasi-isometry type.

# A topological characterization

## Theorem (Gromov)

*If $G$, $H$ are f.g. groups, then the following are equivalent.*

- *$G$ and $H$ are quasi-isometric.*
- *There exists a locally compact space $X$ on which $G$, $H$ have commuting proper actions via homeomorphisms such that $X/G$ and $X/H$ are both compact.*

## Definition

*The action of the discrete group $G$ on $X$ is proper if for every compact subset $K \subseteq X$, the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is finite.*

# Obviously quasi-isometric groups

## Definition

*Two groups $G_1$, $G_2$ are said to be virtually isomorphic, written $G_1 \approx_{VI} G_2$, if there exist subgroups $N_i \leqslant H_i \leqslant G_i$ such that:*

- $[G_1 : H_1], [G_2 : H_2] < \infty$.
- *$N_1$, $N_2$ are finite normal subgroups of $H_1$, $H_2$ respectively.*
- $H_1/N_1 \cong H_2/N_2$.

## Proposition (Folklore)

*If the f.g. groups $G_1$, $G_2$ are virtually isomorphic, then $G_1$, $G_2$ are quasi-isometric.*

# Obviously quasi-isometric groups

## Observation

*If the f.g. groups $G_1$, $G_2$ have isomorphic Cayley graphs with respect to suitable generating sets, then $G_1$, $G_2$ are quasi-isometric.*

## Example (Erschler)

The f.g. groups $\mathrm{Alt}(5)$ wr $\mathbb{Z}$ and $C_{60}$ wr $\mathbb{Z}$ have isomorphic Cayley graphs but are not virtually isomorphic.

## Remark

The quasi-isometry relation is strictly coarser than the transitive closure of the above "obvious quasi-isometries".

## Question

*How many f.g. groups up to quasi-isometry?*

# Growth rates and quasi-isometric groups

## Theorem (Grigorchuk 1984 - Bowditch 1998)

*There are $2^{\aleph_0}$ f.g. groups up to quasi-isometry.*

## Proof (Grigorchuk).

Consider the growth rate of the size of balls of radius $n$ in the Cayley graphs of suitably chosen groups. □

## Proof (Bowditch).

Consider the growth rate of the length of "irreducible loops" in the Cayley graphs of suitably chosen groups. □

# The complexity of the quasi-isometry relation

**Question**

*What are the possible complete invariants for the quasi-isometry problem for f.g. groups?*

**Question**

*Is the quasi-isometry problem for f.g. groups strictly harder than the isomorphism problem?*

# An explicit reduction

Let $S$ be a fixed infinite f.g. simple group. Then the isomorphism problem for f.g. groups can be reduced to the virtual isomorphism problem via the explicit map

$$G \mapsto (\mathrm{Alt}(5) \text{ wr } G) \text{ wr } S$$

in the sense that

$$G \cong H \iff (\mathrm{Alt}(5) \text{ wr } G) \text{ wr } S \approx_{VI} (\mathrm{Alt}(5) \text{ wr } H) \text{ wr } S.$$

## Two Open Questions

- *Does there exist an explicit reduction from the isomorphism problem to the quasi-isometry problem?*
- *Does there exist an explicit reduction from the quasi-isometry problem to the isomorphism problem?*

# What is an explicit map?

**Question**

*Which functions $f : \mathbb{R} \to \mathbb{R}$ are explicit?*

**An Analogue of Church's Thesis**

EXPLICIT $=$ BOREL

- A function $f : \mathbb{R} \to \mathbb{R}$ is Borel if graph($f$) is a Borel subset of $\mathbb{R} \times \mathbb{R}$.
- Equivalently, $f^{-1}(A)$ is Borel for each Borel subset $A \subseteq \mathbb{R}$.

# Polish spaces

## Definition

- *A Polish space is a separable completely metrizable topological space.*
- *E.g.* $\mathbb{R}$, $[0,1]$, $\mathbb{Q}_p$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$,...

## Some less obvious examples

- The space of countable graphs
- The space of finitely generated groups
- etc etc

# The Polish space of countable graphs

- Let $\mathcal{C}$ be the set of graphs of the form $\Gamma = \langle \mathbb{N}, E \rangle$.
- Identify each graph $\Gamma \in \mathcal{C}$ with its edge relation $E \subseteq \mathbb{N} \times \mathbb{N}$.
- Next identify $E$ with the corresponding characteristic function so that $E \in 2^{\mathbb{N} \times \mathbb{N}}$.
- Then $\mathcal{C}$ is a closed subset of the Polish space $2^{\mathbb{N} \times \mathbb{N}}$ and hence is also a Polish space.
- For later use, note that the isomorphism relation on $\mathcal{C}$ is the orbit equivalence relation of the natural action of $\mathrm{Sym}(\mathbb{N})$ on $\mathcal{C}$.

# The Polish space of f.g. groups

- A marked group $(G, \bar{s})$ consists of a f.g. group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators.
- For each $m \geq 1$, let $\mathcal{G}_m$ be the set of isomorphism types of marked groups $(G, (s_1, \cdots, s_m))$ with $m$ distinguished generators.
- Then there exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1_G)).$$

- And $\mathcal{G} = \bigcup \mathcal{G}_m$ is the space of f.g. groups.

# The Polish space of f.g. groups

- Let $(G, \bar{s}) \in \mathcal{G}_m$ and let $d_S$ be the corresponding word metric. For each $\ell \geq 1$, let

$$B_\ell(G, \bar{s}) = \{g \in G \mid d_S(g, 1_G) \leq \ell\}.$$

- The basic open neighborhoods of $(G, \bar{s})$ in $\mathcal{G}_m$ are given by

$$U_{(G,\bar{s}),\ell} = \{\, (H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s}) \,\}, \qquad \ell \geq 1.$$

**Example**

For each $n \geq 1$, let $C_n = \langle g_n \rangle$ be cyclic or order $n$. Then:

$$\lim_{n \to \infty} (C_n, g_n) = (\mathbb{Z}, 1).$$

# A slight digression ...

## Some Isolated Points

- Finite groups
- Finitely presented simple groups

## The Next Stage

- $SL_3(\mathbb{Z})$

## Question (Grigorchuk)

*What is the Cantor-Bendixson rank of $\mathcal{G}$?*

# Borel reductions

## Definition

*Let $E$, $F$ be equivalence relations on the Polish spaces $X$, $Y$.*

- *$E \leq_B F$ if there exists a Borel map $\varphi : X \to Y$ such that*

$$x \, E \, y \iff \varphi(x) \, F \, \varphi(y).$$

  *In this case, $f$ is called a Borel reduction from $E$ to $F$.*

- *$E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$.*
- *$E <_B F$ if both $E \leq_B F$ and $E \nsim_B F$.*

# Borel equivalence relations

**Definition**

*An equivalence relation $E$ on a Polish space $X$ is Borel if $E$ is a Borel subset of $X \times X$.*

**Some Examples**

The following are Borel equivalence relations on the space $\mathcal{G}$ of finitely generated groups:

- the isomorphism relation $\cong$
- the virtual isomorphism relation $\approx_{VI}$
- the quasi-isometry relation $\approx_{QI}$

# The Isomorphism Relation

## Definition

*A Borel equivalence relation $E$ is countable if every $E$-class is countable.*

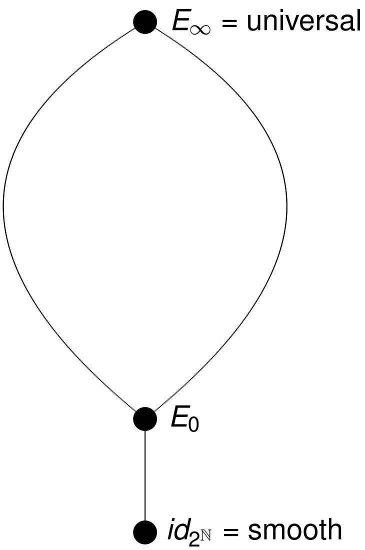
## Observation

*The isomorphism relation $\cong$ is a countable Borel equivalence relation.*

## Proof.

There are only countable many ways to realize a f.g. group $G$ as a marked group $(G, \bar{s})$. $\square$

# Countable Borel equivalence relations



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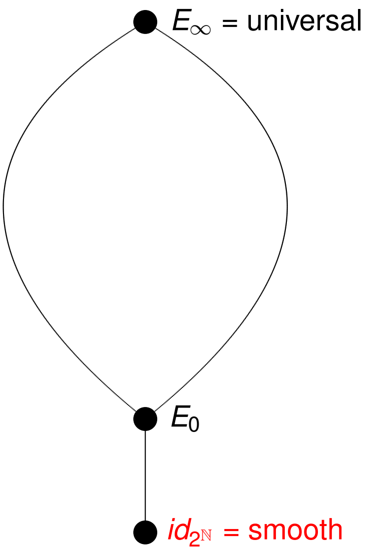

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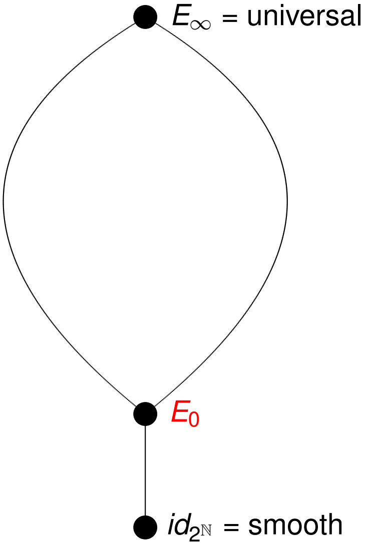
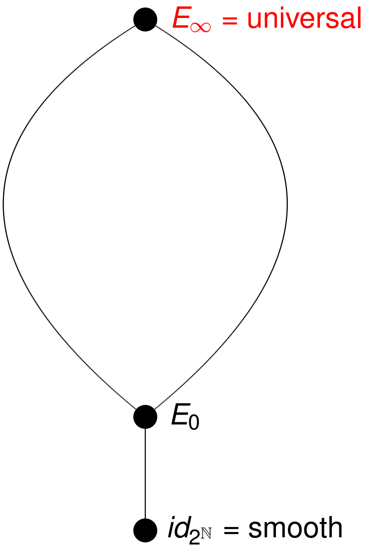



**Definition (HKL)**

*$E_0$ is the equivalence relation of eventual equality on the space $2^{\mathbb{N}}$ of infinite binary sequences.*

# Countable Borel equivalence relations

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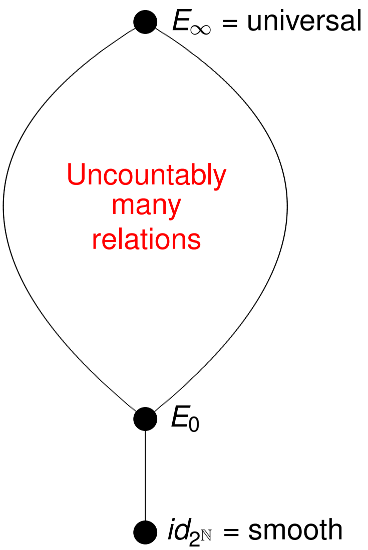
### Definition (DJK)

*A countable Borel equivalence relation $E$ is universal if $F \leq_B E$ for every countable Borel equivalence relation $F$.*

# Countable Borel equivalence relations

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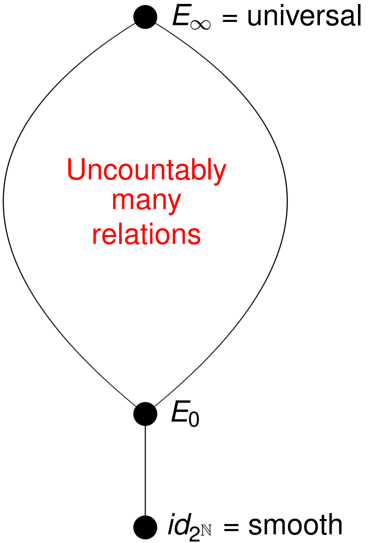
**Question**

*Where does $\cong$ fit in?*

# Countable Borel equivalence relations




Confirming a conjecture of Hjorth-Kechris ...

**Theorem (S.T.-Velickovic)**

$\cong$ *is a universal countable Borel equivalence relation.*

# The Virtual Isomorphism Relation

## Definition

*The f.g. groups $G_1, G_2$ are virtually isomorphic, written $G_1 \approx_{VI} G_2$, if there exist subgroups $N_i \leqslant H_i \leqslant G_i$ such that:*

- $[G_1 : H_1], [G_2 : H_2] < \infty$.
- *$N_1, N_2$ are finite normal subgroups of $H_1, H_2$ respectively.*
- $H_1/N_1 \cong H_2/N_2$.

## Theorem (S.T.)

*The virtual isomorphism problem for f.g. groups is strictly harder than the isomorphism problem.*

# A canonical obstruction

## Definition

*$E_1$ is the Borel equivalence relation on $[0,1]^{\mathbb{N}}$ defined by*

$$x \; E_1 \; y \iff x(n) = y(n) \text{ for almost all } n.$$

## Theorem (Kechris-Louveau)

- *If $G$ is a Polish group and $X$ is a Borel $G$-space, then $E_1 \not\leq_B E_G^X$.*
- *In particular, $E_1$ doesn't reduce to the isomorphism relation on any class of countable structures.*

## Notation

Here $E_G^X$ is the corresponding orbit equivalence relation.

## Theorem (S.T.)

$E_1 <_B \approx_{VI}$.

# A slight digression ...

## Observation

*The abstract commensurability relation $\approx_C$ on the space $\mathcal{G}$ of f.g. groups is countable Borel. And hence $\approx_C$ is Borel reducible to $\cong$.*

## Open Problem

*Find a "group-theoretic" reduction from $\approx_C$ to $\cong$.*

## Theorem (S.T.)

*There does not exist a Borel reduction $\varphi$ from $\approx_C$ to $\cong$ such that $\varphi(G) \approx_C G$ for all $G \in \mathcal{G}$.*

## Conjecture

*There does not exist a continuous reduction from $\approx_C$ to $\cong$.*

# $\mathbf{K}_\sigma$ equivalence relations

## Definition

*The equivalence relation $E$ on the Polish space $X$ is* $\mathbf{K}_\sigma$ *if $E$ is the union of countably many compact subsets of $X \times X$.*

## Example

The following are $\mathbf{K}_\sigma$ equivalence relations on the space $\mathcal{G}$ of finitely generated groups:

- the isomorphism relation $\cong$
- the virtual isomorphism relation $\approx_{VI}$
- the quasi-isometry relation $\approx_{QI}$

# Why is the quasi-isometry relation $\mathbf{K}_\sigma$?

- It is enough to show that each $\approx_{QI} \restriction \mathcal{G}_m$ is $\mathbf{K}_\sigma$.
- For each $\lambda \geq 1$, $C \geq 0$, let $R_{\lambda,C}$ consist of the pairs

$$(\,(\,G, \bar{s}\,), (\,H, \bar{t}\,)\,) \in \mathcal{G}_m \times \mathcal{G}_m$$

  such that there exists a $(\lambda, C)$-quasi-isometry $\varphi : G \to H$.
- If $(\,(\,G, \bar{s}\,), (\,H, \bar{t}\,)\,) \notin R_{\lambda,C}$, then there exists an obstruction in some balls $B_\ell(\,G, \bar{s}\,)$, $B_{\lambda\ell+C}(\,H, \bar{t}\,)$ for some $\ell \geq 1$.
- Thus $R_{\lambda,C}$ is a closed subset of the compact $\mathcal{G}_m \times \mathcal{G}_m$.

# Some universal $\mathbf{K}_\sigma$ equivalence relations

**Theorem (Rosendal)**

*Let $E_{\mathbf{K}_\sigma}$ be the equivalence relation on $\prod_{n \geq 1} \{ 1, \dots, n \}$ defined by*

$$\alpha \, E_{\mathbf{K}_\sigma} \, \beta \iff \exists N \forall k \;\; |\alpha(k) - \beta(k)| \leq N.$$

*Then $E_{\mathbf{K}_\sigma}$ is a universal $\mathbf{K}_\sigma$ equivalence relation.*

**Theorem (Rosendal)**

*The Lipschitz equivalence relation on the space of compact separable metric spaces is Borel bireducible with $E_{\mathbf{K}_\sigma}$.*

# More universal $\mathbf{K}_\sigma$ equivalence relations

## Theorem (S.T.)

*The following equivalence relations are Borel bireducible with $E_{\mathbf{K}_\sigma}$*

- *the growth rate relation on the space of strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$;*
- *the quasi-isometry relation on the space of connected 4-regular graphs.*

## Definition

*The strictly increasing functions $f, g : \mathbb{N} \to \mathbb{N}$ have the same growth rate, written $f \equiv g$, if there exists an integer $t \geq 1$ such that*

- $f(n) \leq g(tn)$ *for all* $n \geq 1$, *and*
- $g(n) \leq f(tn)$ *for all* $n \geq 1$.

# The quasi-isometry vs. virtual isomorphism problems

## The Main Conjecture

*The quasi-isometry problem for f.g. groups is universal* $\mathbf{K}_\sigma$*.*

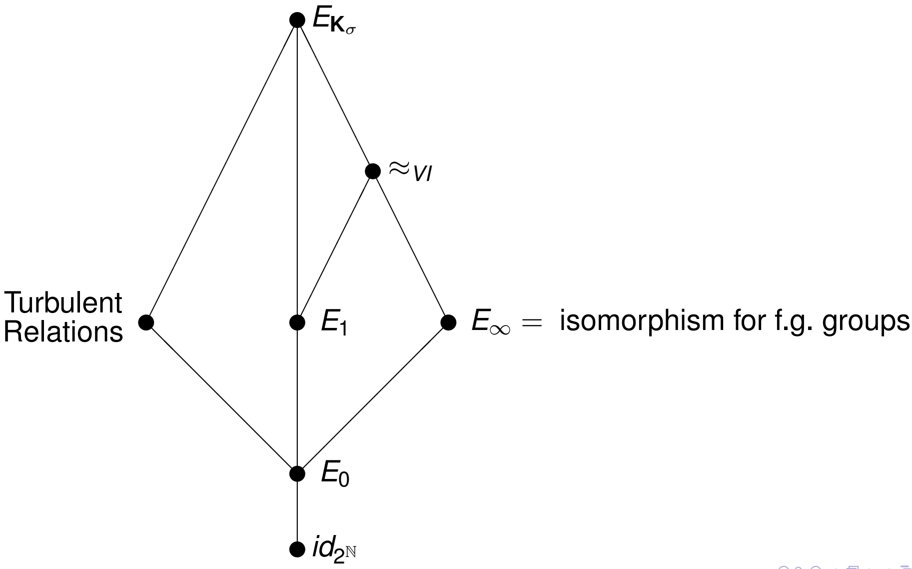
## Theorem (S.T.)

*The virtual isomorphism problem for f.g. groups is not universal* $\mathbf{K}_\sigma$*.*

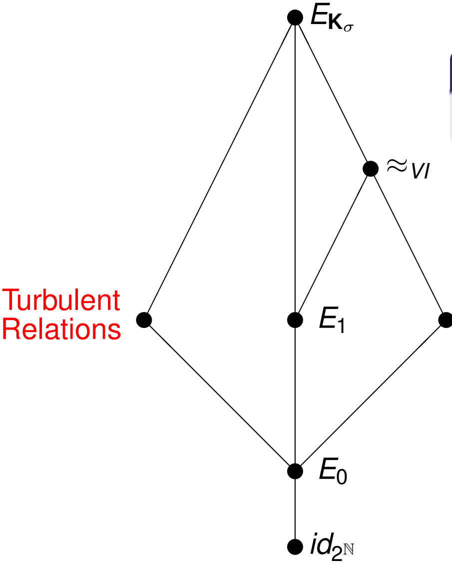
## Corollary

*The virtual isomorphism problem for f.g. groups is strictly easier than the quasi-isometry relation for connected 4-regular graphs.*

# $\mathbf{K}_\sigma$ equivalence relations



# $\mathbf{K}_\sigma$ equivalence relations

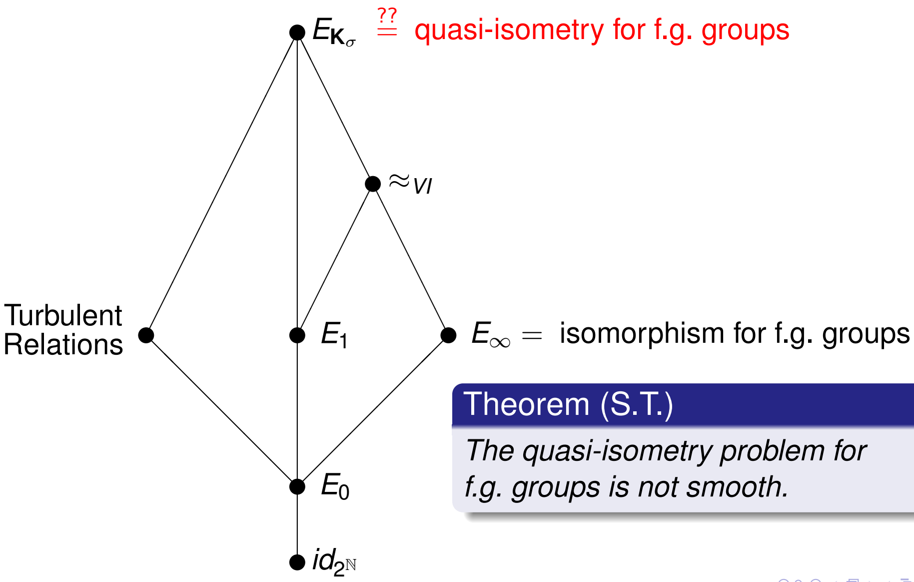



**Theorem (S.T.)**

*If $E$ is turbulent, then $E \not\leq_B \approx_{VI}$.*

$E_\infty =$ isomorphism for f.g. groups

# $\mathbf{K}_\sigma$ equivalence relations




**Theorem (S.T.)**

*The quasi-isometry problem for f.g. groups is not smooth.*

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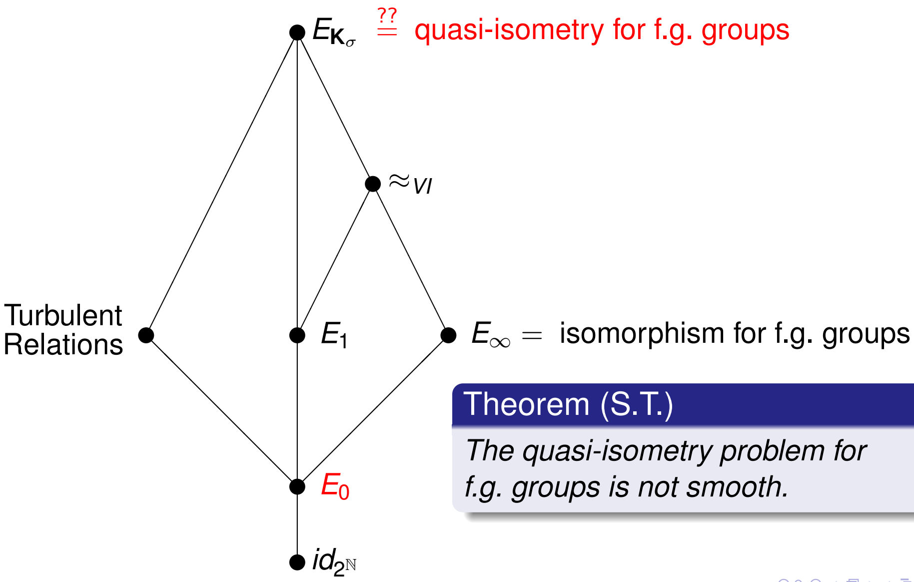


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