The ground state transformation in QFT

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Social background

In July, 1925 on a train ride to Hannover, Born met Pauli, his former assistant and invited him to join him in the further exploration of Born's finding (that Heisenberg's peculiar multiplication rule was just matrix multiplication). But Pauli refused, saying
" You are fond of tedious formalism. You are only going to spoil Heisenberg's physical ideas by your futile mathematics."
When Born got back to Göttingen he asked Jordan to work with him.
Within two months they laid the foundations of matrix mechanics.
(Z f. Phys. submitted September 27, 1925)
Their paper was based on analyzing harmonic oscillators.
In this lecture I'm going to describe the Hilbert state space for harmonic oscillators et al.

Classical harmonic oscillators. One oscillator

$$F = ma \tag{1}$$
$$F = -ky \tag{2}$$
$$-ky = m\ddot{y} \tag{3}$$
$$\ddot{y} = -\omega^2 y, \qquad \omega^2 = k/m \tag{4}$$

$$\text{Potential energy:} \quad V(y) = (1/2)\omega^2 y^2 \tag{5}$$

because

$$\text{Force} \quad = -\omega^2 y = -(d/dy)V(y) \tag{6}$$

on the right side of (4).

$$\textbf{Total energy:} \quad E = (1/2)v^2 + V(y) \quad (m = 1 \quad \text{always}) \tag{7}$$

The configuration space is $\mathbb{R}$ because a point in $\mathbb{R}$ gives the instantaneous position of the point mass.

$$\textbf{Configuration space: } \mathcal{C} = \mathbb{R}. \tag{8}$$

n harmonic oscillators

$$\ddot{y}(t) = -A\, y(t), \quad y(t) \in \mathbb{R}^n, \tag{9}$$

$$A > 0 \quad \text{symmetric, } n \times n \text{ matrix} \tag{10}$$

The natural frequencies are the eigenvalues of $\sqrt{A}$.

$$\textbf{Total energy:} \quad (1/2)|\dot{y}|^2 + (1/2)(Ay, y) \tag{11}$$

$$\textbf{Configuration space:} \quad \mathcal{C} = \mathbb{R}^n \tag{12}$$

The wave equation. Inifinitely many hmc. oscs.

The wave equation over $\mathbb{R}^3$ is

$$\ddot{\varphi}(x,t) = \Delta\varphi(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \qquad \varphi : \mathbb{R}^4 \to \mathbb{R} \tag{13}$$

Classical mechanical rewrite: Let

$$\mathcal{C} = Re\ L^2(\mathbb{R}^3, dx), \qquad A = -\Delta \tag{14}$$

Then (13) can be written

$$\ddot{\varphi}(t) = -A\ \varphi(t), \qquad t \in \mathbb{R}, \quad \varphi(t) \in \mathcal{C} \tag{15}$$

This looks just like (9) with the configuration space $\mathcal{C} = \mathbb{R}^n$ replaced by our new (infinite dimensional) configuration space is

$$\textbf{Configuration space:} \quad \mathcal{C} = Re\ L^2(\mathbb{R}^3, dx). \tag{16}$$

$$\textbf{Total energy:} \quad E = (1/2)\|\dot{\varphi}\|^2_{L^2(\mathbb{R}^3)} + (1/2)(-\Delta\varphi, \varphi) \tag{17}$$

Philosophical statement: The wave equation is just an assembly of infinitely many harmonic oscillators.

Summary of classical harmonic oscillators

One harmonic oscillator.

$$\ddot{y} = -\omega^2 y. \quad \mathcal{C} = \mathbb{R}. \tag{18}$$

n harmonic oscillators.

$$\ddot{y} = -Ay, \quad A > 0, \quad \mathcal{C} = \mathbb{R}^n. \tag{19}$$

The wave equation

$$\ddot{\varphi}(t) = \Delta\varphi(t). \qquad \mathcal{C} = Re\, L^2(\mathbb{R}^3, dx). \tag{20}$$

Step 1. Take configuration space $\mathcal{C}$ and replace its phase space $T^*(\mathcal{C})$ by $L^2(\mathcal{C}, \mu)$, where μ is a wisely chosen measure. $L^2(\mathcal{C}, \mu)$ is the quantum state space.

$$\text{Step 1.} \qquad \text{REPLACE} \quad T^*(\mathcal{C}) \ \text{ by } \ L^2(\mathcal{C}, \mu) \tag{21}$$

Step 2. Replace important functions on $T^*(\mathcal{C})$ (e.g. position, momentum, energy) by corresponding important operators on $L^2(\mathcal{C}, \mu)$.

$$\text{Step 2.} \qquad \text{REPLACE important functions on } \ T^*(\mathcal{C}) \tag{22}$$

$$\text{by important operators on } \ L^2(\mathcal{C}, \mu) \tag{23}$$

Example: One harmonic oscillator

Since $\mathcal{C} = \mathbb{R}$ we can take $\mathcal{H} = L^2(\mathbb{R}, dy)$. Important functions and operators:

$$
\begin{aligned}
\text{Important functions on } T^*(\mathcal{C}) &\rightsquigarrow \text{ operators on } L^2(\mathcal{C}) \\
\text{Position } y &\rightsquigarrow \text{ multiplication by } y \text{ on } \mathcal{H} \\
\text{momentum } p &\rightsquigarrow id/dy \text{ on } \mathcal{H} \\
\text{kinetic energy } (1/2)p^2 &\rightsquigarrow (1/2)(id/dy)^2 = -(1/2)d^2/dy^2 \\
\text{Potential energy } V(y) = (1/2)\omega^2 y^2 &\rightsquigarrow \text{multiplication by } V(y) \\
\text{total energy } (1/2)p^2 + V(y) &\rightsquigarrow -(1/2)d^2/dy^2 + V(y)
\end{aligned}
$$

From all this we only need, for today, the Hamiltonian operator

$$
H = -(1/2)d^2/dy^2 + (1/2)\omega^2 y^2 \quad \text{acting in } L^2(\mathbb{R}, dy) \tag{24}
$$

Before going on to n harmonic oscillators let's write this in a more complicated form. Let

$$\psi(y) = (\omega/\pi)^{1/4} e^{-(1/2)\omega y^2} \tag{25}$$

Facts:

$$1. \quad \|\psi\|^2_{L^2(\mathbb{R})} = 1 \tag{26}$$

$$2. \quad H\psi = (\omega/2)\psi \tag{27}$$

$$3. \quad (\omega/2) \text{ is the smallest eigenvalue of } H \tag{28}$$

Terminology: ψ is called the ground state. (Reasonable?)
Ground state transformation: Let

$$\mu = \psi(y)^2 dy \tag{29}$$

Then

$$\mu(\mathbb{R}) = 1 \tag{30}$$

We are going to change from Hilbert state space $L^2(\mathbb{R}, dy)$ to $L^2(\mathbb{R}, \mu)$.

Since the isomorphisms of quantum mechanics are unitary operators on Hilbert spaces this is legal as long as we implement the change by a unitary operator. Define

$$(Uf)(y) = f(y)/\psi(y), \quad y \in \mathbb{R} \tag{31}$$

Then

$$U : L^2(\mathbb{R}, dy) \to L^2(\mathbb{R}, \mu) \tag{32}$$

is unitary. (10 seconds to verify this.) Fact:

$$UHU^{-1} = (1/2)d^*d + (1/2)\omega \ \text{ acting in } \ L^2(\mathcal{C}, \mu) \tag{33}$$

Digression on the definition of d^*d.

Let M be a Riemannian manifold and let μ be a smooth measure on M. Then the adjoint of d is defined relative to this data by

$$\int_M \langle df, \alpha \rangle d\mu = \int_M \langle f, d^*\alpha \rangle d\mu \tag{34}$$

Here f is a 0-form (function) and α is a 1-form.
If μ happens to be the Riemann-Lebesgue measure associated to the Riemannian metric on M then d^* is the usual coderivative. But for our purposes μ will not be the Riemann-Lebesque measure.

Who needs that constant?

So up to unitary equivalence we have

$$H = (1/2)d^* d + (1/2)\omega \ I_{\mathcal{H}} \qquad \text{acting on} \ \ L^2(\mathcal{C}, \mu) \tag{35}$$

But since the potential energy and therefore total energy is defined only up to a constant we can take the Hamiltonian to be simply

$$H = (1/2)d^* d \qquad \text{acting in} \ \ L^2(\mathcal{C}, \mu), \quad \mathcal{C} = \mathbb{R} \tag{36}$$

without changing the physics.

Example: n harmonic oscillators

We can take over the method from one harmonic oscillator easily. Here is the result:

$$\mathcal{C} = \mathbb{R}^n \quad (37)$$

$$\ddot{y} = -Ay, \quad A > 0 \quad (38)$$

$$\text{Define } \mathcal{H} = L^2(\mathbb{R}^n, \mu) \quad (39)$$

$$d\mu(y) = C \exp(-(\sqrt{A}\, y, y)_{\mathbb{R}^n}) d^n y \quad (40)$$

And the Hamiltonian in the ground state representation is

$$H = (1/2) d^* d + \frac{1}{2} trace \sqrt{A} \;\; I_{\mathcal{H}} \quad (41)$$

We could drop the last term if we wished because changing energy by an additive constant doesn't change the physics

Wave equation

We need only parrot all the preceding computations and procedures. Everything is the same except that $\mathbb{R}^n$ is now replaced by $Re\ L^2(\mathbb{R}^3, dx)$, which is infinite dimensional. Here is the result. We can take the quantum Hilbert space to be

$$\mathcal{H} = L^2(\mathcal{C}, \mu), \quad \mathcal{C} = Re\ L^2(\mathbb{R}^3, dx) \tag{42}$$

$$\mu = const. \exp(-\sqrt{(-\Delta)}\ \varphi, \varphi)_{\mathcal{C}})\mathcal{D}\varphi \tag{43}$$

$$H = (1/2)d^*d + \frac{1}{2} trace \sqrt{(-\Delta)}\ \ I_{\mathcal{H}} \tag{44}$$

As before we can drop the (now) infinite constant $(1/2) trace(\sqrt{(-\Delta)})$ to find

$$H = (1/2)d^*d \quad \text{acting in} \quad L^2(\mathcal{C}, \mu) \tag{45}$$

Drop infinite constant?

OK with you to drop the infinite constant? No? Let me read to you from a QFT book.
(1962) " We drop the infinite constant so that the no-particle state has zero energy"

OK? No? Let me read to you from a more recent book.

(1965) " This is the first of a number of divergences that we shall encounter in field theory. It is the easiest one to remove simply by subtracting off an infinite constant. This can be done because (...) only energy differences have a physical meaning."

OK? Still not OK? Let me read to you from an even more recent book.

(1979) " This is an example of a spurious difficulty arising from too literal an interpretation of the Correspondence Principle."

OK? Good.

Yang-Mills fields

The current view of classical fields in elementary particle theory is this.

1. A **matter field** is a section of a vector bundle over $\mathbb{R}^4$ (Minkowski spacetime)

2. A **force field** is a connection on this vector bundle.

Let's just focus on the force field.

Notation: K a compact Lie group. $\mathfrak{k} = Lie(K)$. K is to be determined by experiment. For example $K = SU(3) \times SU(2) \times U(1)$ is the currently accepted "right" group.

A connection is determined by an equivalence class of connection 1-forms $A(x, t)$ with values in $\mathfrak{k}$. Two such $\mathfrak{k}$ valued 1-forms are equivalent if there is a "gauge function" $g : \mathbb{R}^4 \to K$ such that $A^g := g^{-1}Ag + g^{-1}dg$ is the other 1-form. Time evolution of a gauge field (i.e. the analog of the wave equation above) is

$$D_A^* F = 0 \quad \text{(Yang-Mills hyperbolic equation)} \tag{46}$$

where F is the curvature of the connection form A and D_A^* is the coderivative of the gauge covariant exterior derivative D_A. (The Minkowski metric enters into the definition.)

Quantization of Yang-Mills fields

As usual we need first to spell out what the configuration space is. Here it is

$$\mathcal{C} = \frac{\text{the set of all } \mathfrak{k}\text{-valued 1-forms on } \mathbb{R}^3}{\text{the gauge group of } \mathbb{R}^3} \tag{47}$$

Semi-Theorem 1. There exists a finite measure μ on $\mathcal{C}$ and a unitary representation U of the inhomogeneous Lorentz group $\mathcal{L}$ on $L^2(\mathcal{C}, \mu)$ such that

$$(1/2)d^*d = idU(d/dt) \quad (= iU_*(d/dt)). \tag{48}$$

Moreover, for each closed curve $C \subset \mathbb{R}^3$, parallel transport around C

$$W_C(A) := \chi \circ //_C^A \quad (\chi \text{ is any character of } K) \tag{49}$$

defines a function on $\mathcal{C}$ such that $\{W_C(\cdot) : C \subset \mathbb{R}^3\}$ is fundamental in $L^2(\mathcal{C}, \mu)$.

Proof of Semi-theorem 1

Proof is nowhere in sight.

Definition: Semi-theorem. A semi-theorem is a theorem whose statement is at most half true and for which at most half of its proof exists.

Final semi-theorem

Semi-Theorem 2 (reward for proof = \$1M) Take

$$\mathcal{C} = \frac{\{\mathfrak{k} \text{ valued connection forms on } \mathbb{R}^3\} \times L^2(\mathbb{R}^3; \mathfrak{k})\}}{\text{modulo gauge group}} \tag{50}$$

There exists a finite measure μ on $\mathcal{C}$ such that

$$\text{inf spectrum } (d^* d | 1^{\perp}) > 0. \tag{51}$$

a. The measure μ should be the ground state measure for the Higg's field (whose configuration space is the second factor in (50)).

b. The Lorentz group should act unitarily, as usual and $(1/2)d^* d$ should be the infinitesimal generator of time translations.

Pauli to Dirac, 1932

Lest you think that the discovery of QFT followed some kind of straight and logical line, here is a letter from Pauli to Dirac (shortened by me) 1932

"Your recently published remarks in the Proc. of the Royal Society concerning quantum electrodynamics was certainly no masterpiece. After a confused introduction, that consisted of only half understandable sentences, you come finally to results in a simplified one -dimensional example that Heisenberg and I already found by our formalism This stands in marked contrast with your assertion in your introduction that your QED is better than ours.