Cornell Topology Festival 2018:
Panel Discussion

This is a report on the panel discussion of the 54^{th} Topology Festival at Cornell University, which ran from May 11^{th} to May 13^{th}, 2018. Each of the speakers was given approximately 5 minutes to outline either a particularly interesting recent result, or an open problem in the field. Summaries of their presentations follow.

Reported by: Kimoi Kemboi, Yun Liu, Elise McMahon, and John Whelan

A Fast Algorithm to Classify Elements of Mapping Class Groups

Tara Brendle, University of Glasgow

Let S_g a compact orientable surface, and $f \in \text{Mod}(S_g))$, the mapping class group of S_g. Then we have by the Nielsen-Thurston classification that f is one of the following:

(1) finite order
(2) reducible
(3) pseudo-Anosov

Additionally, $\text{Mod}(S_g)$ is finitely generated (by Dehn twists, for example).

Question: given $f \in \text{Mod}(S_g)$, written as a product of generators, how do we determine which of (1)-(3) f corresponds to?

Margalit, Strenner, and Yurttas recently produced an algorithm which is quadratic in the word length of f. This algorithm determines whether f is (1), (2), or (3), as well as the order(1), reducing curves(2), or foliations and stretch factors(3) of f. Previously known algorithms were exponential.

The general idea of the new innovation is as follows: given f, iterate a certain number of times (chosen based on the genus g) to get into a piecewise linear region of the manifold. Then f raised to that power can be represented by a matrix, and using an algorithm of Toby Hall's the foliations and stretch factors can be obtained.

The Chas-Sullivan Lie Bracket

Vladimir Turaev, Indiana University

Let M a smooth n-dimensional manifold, and $\mathcal{L}M = \{f : S^1 \to M | f \text{ is continuous}\}$ the loop space of M. Letting S^1 act on $\mathcal{L}M$ in the evident way, we obtain the equivariant homology H_* of this action (roughly) by defining H_* to be the homology of $\mathcal{L}M$ modulo the action of S^1. Chas and Sullivan defined a Lie Bracket in H_*.

In general $[H_i, H_j] \subset H_{i+j+2-n}$.

Their question is: What information about M is contained in H_* with the Lie bracket?

One result (Chas): When M is a 3-manifold, we can recover the geometric classification of the manifold.

The Stable Homology of Mapping Class Groups

Thomas Church, Stanford University

This is work of Jeremy Miller, Peter Patzt, and Jennifer Wilson.

When g is much, much larger than i, we get that $H_i(Mod_g; \mathbb{Z}) \cong H_i(Mod_g; \mathbb{Z})$.

In that vein, to what extent can we talk about the *stable homology* of Torelli_g or IA_n? Well, the problem is that we know it doesn't stabilize.

The symplectic group $Sp_{2g}\mathbb{Z}$ acts on $H_i(\text{Torelli}_g)$, and similarly $GL_n\mathbb{Z}$ acts on $H_i(IA_n)$. We can use these to approach the problem with representation theory.

Theorem (MPW): $H_2(IA_n; \mathbb{Z})$ is centrally stable as $GL_n\mathbb{Z}$-representations for $n \geq 38$.

This is the first time that people have used this kind of representional stability for these types of infinite groups.

$\mathbf{H}_*$ and $\mathbf{HR}_*$

Ajay Ramadoss, Indiana University

Question: How's H_* as defined above related to HR_*?
The motivation begins with the work by Susanna Fishel, Ian Grojnowski, and Constantin Teleman about the strong Macdonald conjecture.
Let $\mathfrak{g}$ be a reductive Lie algebra with Cartan subalgebra $\mathfrak{h}$, with Weyl group W and root system R, one has the q-MacDonald identity for any k,

$$\frac{1}{|W|}\prod_{j=1}^{k-1}\left(1-q^j\right)^l CT\left\{\prod_{j=0}^{k-1}\prod_{\alpha\in R}\left(1-q^j e^{\alpha}\right)\right\}=\prod_{i=1}^{l}\prod_{j=1}^{k-1}\left(1-q^{km_i+j}\right)$$

where m_i are the exponents of $\mathfrak{g}$. The left hand side is the Euler characteristic of the relative Chevalley-Eilenberg complex $\mathcal{C}\left(\mathfrak{g}\left[z\right]/z^k,\mathfrak{g};\mathbb{C}\right)$, and this identity explicitly computes $H_{\bullet}\left(\mathfrak{g}\left[z\right]/z^k,\mathfrak{g};\mathbb{C}\right)$ as a free bigraded commutative algebra and specifies that weights and homological degrees of a set of homogeneous generators. Therefore one has an isomorphism of graded commutative algebras

$$HR_*\left(\mathbb{CP}^r,G\right)^G\cong\bigwedge_k\left(\xi_1^{(i)},\xi_3^{(i)},\cdots,\xi_{2r-1}^{(i)}:i=1,2,\cdots,l\right),$$

where $\xi_{2s-1}^{(i)}$ has homological degree $2rm_i+2s-1$, an it turns out that this isomorphism has a natural topological interpretation, it is given by a canonical algebra map

$$\mathrm{Sym}_k\left(\bigoplus_{i=1}^{l}\bar{H}_*^{\mathbb{S}^1,(m_i)}\left(\mathcal{L}\mathbb{CP}^r\right)\right)\to HR_*\left(\mathbb{CP}^r,G\right)^G$$

called the Drinfeld homomorphism. The domain is the graded symmetric algebra of the direct sum of certain (topologically defined) eigenspaces of the $\mathbb{S}^1$-equivariant homology of the free loop space $\mathcal{L}\mathbb{CP}^r$. The Drinfeld homomorphism exists for any simply-connected space X, and it is an isomorphism in the following cases: $X=\mathbb{S}^n,(n\geq 2)$, $X=\mathbb{CP}^r,(r\geq 2)$ and $X=\mathbb{S}^{2n+1}\times\mathbb{CP}^{\infty},(n\geq 1)$. An interesting question is for what kind of loop spaces can such a homomorphism exist?

A Simple Definition of Topological Cyclic Homology

Brooke Shipley, University of Illinois at Chicago

This is a work by Thomas Nikolaus and Peter Scholze in their paper 'On topological cyclic homology' in 2017. They gave another construction of topological cyclic homology which is much simpler and easier to understand. Topological cyclic homology is a refinement of Connes' cyclic homology which was introduced by Bökstedt–Hsiang–Madsen in 1993 as an approximation to algebraic K-theory. To define TC, we need topological Hochschild homology, and define cyclotomic spectra, and then we can define TC from the cyclotomic structure on THH.

A cyclotomic spectrum is a spectrum X with $\mathbb{T}$-action together with $\mathbb{T}$-equivariant maps $\phi_p : X \to X^{tC_p}$ for every prime p.

The definition of TC is given by

$$TC\,(A)_p^n \cong \mathrm{eq}\left\{ THH\,(A)^{h\mathbb{T}} \underset{\text{can}}{\overset{\mathcal{O}_p}{\rightrightarrows}} \left(THH\,(A)^{tCp}\right)^{h\mathbb{T}} \right\}_p$$

where THH has a S^1-action together with S^1-equivariant maps $\mathcal{O}_p : THH\,(A) \to THH\,(A)^{tCp}$ for every prime p, and we denote S^1 by $\mathbb{T}$.

Elliptic Curves and Skein Algebras

David Jordan, University of Edinburgh

This question arises from a surprising connection between elliptic curves over $\mathbb{F}_q$ and skein algebras (specifically the HOMFLYPT), which was determined by Burhan-Schiflmann. Consider an elliptic curve E over a finite field and the category of coherent sheaves over E, $coh(E)$. One can define the Hall algebra given by isomorphism classes of $\mathcal{F}$,$Ha(Coh(E)) =$ $[\mathcal{F}]$, where $\mathcal{F}$ is a sheaf. There is a product $[\mathcal{F}] \cdot [\mathcal{G}] = \sigma m[\mathcal{H}]$, where m are polynomials in Frobenius eigenvalues of E. This counts extensions by $\mathcal{H}$, i.e. short exact sequences of the form $0 \to \mathcal{F} \to \mathcal{H} \to \mathcal{G} \to 0$.

Morton-Samuelson, then showed that the skein algebra of the torus T^2, which is a a vector space spanned by links and $T^2 \times I$ modulo skien relations and denoted $sk(T^2)$, is isomorphic to $\simeq E_{q,q}$. A question that one may ask is what skein algebra can be used to obtain these algebras A_q, t.

Soft Descriptions of Number Fields

Mikhail Khovanov, Columbia University

There is an analogy between number fields and 3-manifolds. One can find an instance of this analogy for example in the book *Knots and Primes* by Morishita. This analogy is at the homological level in the sense that if you take a 3-manifold M and consider the category of sheaves of M on vector spaces, it has homological dimension 3 and on the other hand, there are some homology theories that can be built out of number fields. While these seem like naive analogies, there is something deeper in this correspondence.For example, for 3-manifolds, we have two general descriptions, the "soft description", namely that 3-manifolds can be triangulated, and the "rigid" description, that is, with some exceptions, they have a hyperbolic structure. In the case of number fields, however, we have only the “rigid” description given by its ring of integers. One insteresting problem is to find, with appropriate assumptions, a “soft” description of number fields. More precisely, to find a way to build a number field out of local structures that are less rigid in the same way that 3-manifolds can be built out of tetrahedra.

Volume Conjectures for Quantum Invariants

Helen Wong, Carleton College

For each quantum invariant, there is a volume conjecture.

Link Invariants:
Begin with the Jones polynomial for a link $L \subset \mathbb{S}^3$.
Can generalize to the n-th Jones polynomial for $L \subset \mathbb{S}^3$.

For 3-Manifolds:
Turaev-Viro invariant: Take a triangulation, color the vertices, edges, and triangles, and compute using a well-defined formula.
Witten-Reshetikhin-Turaev invariant: Can obtain M^3 by surgery from framed link S^3, coloring with linear combination of Jones polynomials.

Kashaev (1995), and reformulated by Murakami-Murakami (2000) conjectured that if L is hyperbolic, then

$\lim_{N\to\infty} \frac{2\pi}{N} \log |J_n(L, e^{2\pi i/N})| = \mathrm{Vol}(\mathbb{S}^3 \backslash \mathrm{L})$.

Similar volume conjectures exist for the Turaev-Viro and the Witten-Reshetikhin-Turaev invariants. These conjectures have been checked for a small number of knots, but the full conjectures are believed to be wide open.

Determining Whether Homeomorphisms are Isotopic

Liat Kessler, Cornell University

Question: If $\mathcal{CP}^2$ is blown up 6 times, is it simplectically isotopic to the identity? Towards this, what are some methods that mathematicians use to approach the question of showing that something is isotopic to the identity or even in the same component as the identity? In other words, how does one tell if homeomorphisms or diffeomorphisms are isotopic?

A few answers from the audience: for hyperbolic manifolds, two homeomorphisms are isotopic if and only if they have the same fundamental group. For surfaces, a homeomorphism is isotopic to the identity if one can find a collection of curves that cuts the surface into disks such that the homeomorphism fixes those curves.