Cornell Topology Festival 2022
Panel Discussion

May 7, 2022

This is a report on the panel discussion of the 57th Topology Festival at Cornell University, which ran from May 6 to May 8, 2022. Each of the speakers was given approximately 5 minutes to outline either a particularly interesting recent result, or an open problem in the field. Summaries of their presentations follow.

Reported by: Isaac Goldberg, Nicki Magill, David Mehrle, Nikhil Sahoo, Kimball Strong, Chase Vogeli.

Path Induction in Homotopy Type Theory
Emily Riehl, Johns Hopkins University

Homotopy type theory is an exciting new field that provides a collection of proof techniques that hold in any ∞-topos. One of the surprising techniques is called path induction, which is analogous to the induction principle for natural numbers.

For a space A and a point $a \in A$, we can form the based path space P_aA by the pullback diagram

$$\begin{tikzcd} P_aA \arrow[r] \arrow[d, two heads, "\sim"] & A^I \arrow[d, two heads, "\sim"] \\ 1 \arrow[r] & A, \end{tikzcd}$$

which is contractible. The constant path is the basepoint of P_aA, but in homotopy type theory it goes by the name 'refl_a'.

The map $1 \to P_aA$ is an acyclic cofibration since P_aA is contractible, and hence lifts on the left against any fibration. So given a fibration $P \twoheadrightarrow P_aA$, there is a lift

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, hook, "\mathsf{refl}_a"', "\sim"] & P \arrow[d, two heads] \\ P_aA \arrow[r, equal] \arrow[ur, dashed] & P_aP \end{tikzcd}$$

and this lifting property enriches as follows: the map $Map_{P_aA}(P_aP, P) \to P_{\mathsf{refl}_a}$ is a trivial fibration. Now we can state the main theorem.

Theorem. *For any fibration $P \twoheadrightarrow P_aA$ over the based path space, the map below has a section, which is called path-induction*

$$\begin{tikzcd} Map_{P_aA}(P_aA, P) \arrow[r, "ev_{\mathsf{refl}_a}"] & P_{\mathsf{refl}_a} \arrow[l, dashed, bend left, "\text{path-induction}"] \end{tikzcd}$$

This can be rephrased as the following theorem in homotopy type theory.

Theorem. *For any family of types P depending on a point $x \in A$ and a path p from a to x, to prove*

$$\prod_{x:A} \prod_{p:a=x} P(x, p),$$

it suffices to prove the case x is a and p is refl_a.

Holomorphic Floer Theory

Dusa McDuff, Barnard College

There is new exciting progress in symplectic geometry by Abouzaid, Blumberg, McLean and Smith called holomorphic Floer theory. This works for genus zero J-holomorphic curves

$$(\Sigma, j) \xrightarrow{hol} (M, J)$$

in some class A mod reparametrizations. There is a compact moduli space of solutions $\bar{\mathcal{M}}$. We want to cut it down to something zero-dimensional and then count the number of things in it. This moduli space isn't amenable to that technique, so people use approximate solution techniques instead. They build local models and get rational counts of the solutions.

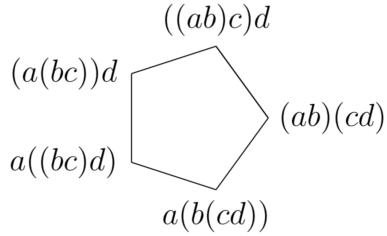
However, the work of Abouzaid–Blumberg and Abouzaid–McLean–Smith has a totally different way of looking at it. They build a new model by taking a quotient of some Lie group. This results in an orbifold, but a much nicer orbifold then in previous work. Namely, it has an almost complex structure, which allows them do some homotopy theory. They use homotopy theory to define invariants over $\mathbb{Z}/p$ instead of working over $\mathbb{Q}$.

Multiplicative Structures in Higher Algebra

Arpon Raksit, Massachusetts Institute of Technology

A multiplication on a set X is a map $X \times X \to X$. To ask for it to be associative means to require $(ab)c = a(bc)$. The situation for spaces can be more subtle. Instead of asking for our multiplication to be strictly associative, we may only ask that there exists a path from $a(bc)$ to $(ab)c$. Having such paths is called A_2-structure.

There are five ways to parenthesize four elements, and once you fix a homotopy from $a(bc)$ to $(ab)c$, you get a pentagon:



Filling in the pentagon with a homotopy between these homotopies yields what is called A_3-structure. There is an infinite hierarchy $A_2, A_3, A_4, \ldots$ of these structures, and their (co)limit A_∞ encodes that your multiplication is as associative as you wish.

For abelian groups, a multiplication on A is a map $A \otimes A \to A$, which gives a ring structure. The analogous object in stable homotopy theory a multiplication on a spectrum S: a map $S \wedge S \to S$.

The fundamental example of a spectrum is the sphere spectrum $\mathbb{S}$, which plays an analogous role in stable homotopy theory to that of the integers $\mathbb{Z}$ in commutative algebra. It's built out of the spheres $S^0, S^1, S^2, \ldots$ and is related to stable homotopy groups of spheres.

Once can assemble pushouts of the form

$$\begin{tikzcd} S^n \arrow[r, "2"] \arrow[d] & S^n \arrow[d] \\ D^n \arrow[r] & S^n/2, \end{tikzcd}$$

along degree 2 maps $S^n \to S^n$ to get a quotient $\mathbb{S}/2$ of the sphere spectrum. An old theorem states $\mathbb{S}/2$ admits no unital multiplication, in constrast to the case of $\mathbb{Z}/2$. However, Burklund proved the following just this year.

Theorem. $\mathbb{S}/8$ *admits an* A_∞*-structure.*

The proof uses the new theory of synthetic spectra. In general, the spectra $\mathbb{S}/p^n$ have more multiplicative structure as you increase the prime p and n.

Injectivity Radius

Ian Biringer, Boston College

Let X by higher rank, i.e. $n \geq 3$, irreducible symmetric space equipped with Riemannian metric that is invariant under $SL_n(\mathbb{R})$ action. Think $X = SL_n(\mathbb{R})/SO(n)$. Let $M = X/\Gamma$ where Γ acts on X by isometries and is properly discontinuous. If $p \in M$, define

$$\mathrm{inj}_M(p) = \tfrac{1}{2}(\text{length shortest essential loop in } M \text{ at } p)$$

Then, the main result of Fraczyk–Gelander is

Theorem. $\sup\{inj_M(p) | p \in M\} < \infty$ *if and only if* M *has finite volume.*

Note, if $n = 2$ in the case of $\mathbb{H}^2$, take infinite M with infinite holes and here we have infinite volume, but can find loop with finite radius.

To prove the theorem, they use random walks and ergodic theory.

Translation Surfaces

Aaron Calderon, Yale University

A translation surface can be thought of in two ways, either as a riemann surface together with a holomorphic 1-form, or as a singular euclidean surface with isolated cone points where the angle is a multiple of 2π. The moduli space of all translation surfaces of a given genus decomposes into *strata*. For example $\Omega\mathcal{M}_3(4)$ denotes the moduli space of holomorphic 1–forms with a single zero of order 4 on surfaces of genus 3. Calderon attributed the following informal conjecture to Kontsevich:

Conjecture. *Components of strata are $K(\pi, 1)$ spaces for "some kind of mapping class group".*

Some evidence for the conjecture is that it is true for hyperelliptic components. But it is open in general.

An example: $\Omega\mathcal{M}_3(4)$ is a $K(\pi, 1)$ for the right-angled Artin group based on E_6. There is a natural map

$$\pi_1(\Omega\mathcal{M}_3(4)) \to \mathrm{MCG}_{3,1}$$

Waynryb showed this map has nontrivial kernel by exhibiting an element but without really explaining why there is kernel.

Question. Why is there kernel?

Variations on Configuration Spaces

Jenny Wilson, University of Michigan

For a space X, one can form the ordered configuration space of k points in X:

$$F_k(X) = \{(x_1, \ldots, x_k) : x_i \neq x_j \text{ for all } i \neq j\}.$$

The quotient of $F_k(X)$ by the action of the symmetric group S_k which permutes the k points is the unordered configuration space

$$B_k(X) = \{k\text{-element subsets of } X\}.$$

An important example is the case $X = \mathbb{R}^2$, for which $B_k(\mathbb{R}^2)$ (resp. $F_k(\mathbb{R}^2)$) is a $K(\pi, 1)$ for π the braid group (resp. pure braid group). Indeed, one can visualize loops in these configuration spaces as (pure) braids. These spaces have also been studied in the case when X is a manifold or a graph.

One variation on this line of inquiry is to give X a metric and consider the configuration space of (disjoint) discs of a fixed radius in X. Such problems blend questions of braiding and sphere packing. Alpert–Manin studied this for the strip $\mathbb{R} \times [0, W]$ of width W and computed the cohomology of the resulting configuration space. In-progress work of Wawrykow seeks to understand these cohomology groups as S_k-representations.

Dusa McDuff mentioned that there are related problems concerning configuration spaces of symplectically embedded balls of a fixed radius.

Counting Results on Surfaces

Tarik Aougab, Haverford College

Let S be a surface. A common type of result gives an asymptotic count of objects γ on S. Here, the objects γ could be some type of curve, etc. Lots of these results were inspired by the work of Maryam Mirzakhani. Then the general kind of theorem is:

Theorem. $\#\{\alpha \in MCG \cdot \gamma |\ \ell(\alpha) \leq L\} \sim C_{Top} C_X L^{6g-6}$

where $MCG \cdot \gamma$ denotes orbits under the actions of the mapping class group, C_{Top} is a 'topological constant,' C_X is a 'metric constant,' and the $6g - 6$ comes from Teichmuller space.

One very general form of such a theorem is due to Rafi–Souto, where γ is a geodesic current, also known as a filling current.

Also, everyone should read "Turning Coffee into Unions: Mathematicians and Collective Bargaining" by Denis Hirschfeldt.

Higher Teichmüller theory

Sara Maloni, University of Virginia

Given a surface S, one can study the space of discrete faithful representations of $\pi_1(S)$ into $PSL(2, \mathbb{C})$, up to conjugacy. The interior of this space is the space of *quasi-fuchsian* surface groups $QF(S)$; Teichmüller space sits inside $QF(S)$ in a natural way. The boundary is described in a very satisfying way by the Ending Lamination Theorem. One would like to have a similar theory for different Lie group targets. This is the subject of Higher Teichmüller Theory. If the target is $PSL(d, \mathbb{R})$ there is a natural subset of the space of representations up to conjugacy called the *Hitchin component*. Perturbing into $PSL(d, \mathbb{C})$ we obtain the space of "Quasi-Hitchin" representations, and one can ask about whether the space of such representations has closure describable by something like an Ending Lamination Theorem.

Maloni also described the closely related notion of *extended geometrically finite* representations. Recently Weisman has proved a stability result for these representations.

Crazy Subgroups of Hyperbolic Groups

Genevieve Walsh, Tufts University

Let G be a hyperbolic group. Then G acts geometrically (properly and co-compactly) on some Gromov hyperbolic space X. For example if G is the fundamental group of a hyperbolic 3-manifold M we can take X to be $\mathbb{H}^3$. A subgroup $H < G$ is *quasi-convex* if it acts cocompactly on the convex hull of its limit set in ∂X.

There exist subgroups of hyperbolic 3-manifold groups which are not quasi-convex. For example if M^3 fibers over S^1 and H is the fundamental group of the surface fiber, then H is not quasi-convex. An emerging subfield led by Martelli, Italiano, and others seeks to find other such examples of crazy subgroups of hyperbolic groups. One particular question is whether these exist in one dimension higher:

Question. Does there exist a subgroup $H = \pi_1 M^3$ inside $G = \pi_1 M^4$ where ∂G is S^3 and H is not quasi-convex?