Cornell Topology Festival 2023
Panel Discussion

May 6, 2023

This is a report on the panel discussion of the 58th Topology Festival at Cornell University, which ran from May 5 to May 7, 2023. Each of the speakers was given approximately five minutes to outline a particularly interesting recent result or an open problem in their field. Summaries of their presentations follow.

Reported by: Colby Kelln, Nicki Magill, Nikhil Sahoo, Chaitanya Tappu, Morgan Weiler, Chase Vogeli.

(∞, n)-categories in homotopy type theory
Paige North, University of Pennsylvania

An (∞, n)-category is a category with infinitely many levels of morphisms, in which any morphism above level n is invertible. At the time of its inception, homotopy type theory (HoTT) was hoped to be a framework to understand these higher categories, but this has been difficult to realize.

Some examples of *coherences* that categories need to have are those which encode associativity. Given three composeable morphisms f, g, and h, we have $h \circ (g \circ f) = (h \circ g) \circ f$ (and this equality is encoded by the existence of a morphism one level higher). Given four composeable morphisms f, g, h, and i, we have

$$i \circ (h \circ (g \circ f)) = i \circ ((h \circ g) \circ f) = \cdots ,$$

and so on. One approach for higher categories, in analogy with simplicial sets, could be to study functors $\mathbf{\Delta} \to \mathsf{Type}$, but in HoTT these are ∞-functors, so the coherences are weak themselves and this becomes very difficult/impossible.

Recently, augmentations to homotopy type theory which accomodate higher categories have been proposed. The *two-level type theory* of Annenkov–Capriotti–Kraus–Sattler adds a second notion of strict equality on top of the usual homotopy equivalence in HoTT. The $(\infty, 1)$-*type theory* of Riehl–Shulman adds in some additional syntax that allows one to speak of simplex types Δ^n and use these to identify types modelling higher categories. One question which remains is what the minimal addition to HoTT to enable the study of higher categories is.

Rigidity of hyperbolic manifolds with piecewise totally geodesic boundary

Jean Lafont, Ohio State University

A closed hyperbolic manifold is the quotient $\mathbb{H}^n/\Gamma = M^n$ where Γ is a discrete group of isometries of $\mathbb{H}^n$ without elliptics or parabolics. It is well known that there is a big difference between $n = 2$ and $n \geq 3$. For $n = 2$ and a genus g surface $M^2 = \Sigma_g$, there are many hyperbolic structures on Σ_g parametrized by Teichmüller space, whereas for $n \geq 3$, the hyperbolic structures are rigid by Mostow rigidity, that is, there is a unique hyperbolic structure, which is determined by $\pi_1(M^n)$.

Question: What if we allow manifolds with piecewise totally geodesic boundary?

A result by Frigerio states that if there is a single totally geodesic piece in the boundary, then the hyperbolic manifold is rigid.

We have the following recent result:

Theorem (Gustavo Chaparro-Sumalave)**.** If $n \geq 6$ and if there is a single bend in the manifold then it is topologically rigid. That is, if M_1^n and M_2^n are two hyperbolic manifolds with the same boundary and same fundamental group, then there is a homeomorphism between them.

Topological Versus Smooth Unknotting of Spheres

Robert E. Gompf, University of Texas at Austin

Suppose $K : S^2 \to \mathbb{R}^4$ is a smooth embedding. If K is topologically unknotted, must it also be smoothly unknotted? (Freedman showed that the sphere K being topologically unknotted is equivalent to $\pi_1(\mathbb{R}^4 - K) \cong \mathbb{Z}$, akin to the theorem of Papakyriakopoulos for knots in S^3.) If we only assume K is a locally flat embedding, must it be isotopic to a smooth embedding? These are two examples of how little is known about the distinction between the topological and smooth settings for knotted spheres. Even if we drop the unknottedness assumption, we do not know of any smoothly knotted spheres in $\mathbb{R}^4$ that are topologically equivalent but not smoothly equivalent.

But certain counterexamples are known if we allow more complicated surfaces or 4-manifolds. In 1988, Finashin, Kreck and Viro described embeddings $\#_{10}\mathbb{RP}^2 \hookrightarrow \mathbb{R}^4$ that are topologically equivalent but not smoothly equivalent. Finashin improved this to embeddings $\#_6\mathbb{RP}^2 \hookrightarrow \mathbb{R}^4$ in 2009. They distinguish the smoothly knotted surfaces by comparing two-fold branched covers ramified along the surfaces, which end up being homeomorphic but not diffeomorphic. Might the open questions above be approached using similar techniques?

Minimal Models of Toric Manifolds

Margaret Symington, Mercer University

We can consider the following inclusions:

$$\{\text{toric manifolds}\} \subset \{\text{symplectic manifolds}\} \subset \{\text{smooth manifolds}\}$$

where symplectic manifolds are sitting inside smooth manifolds in a rather "messy way." Real 4-dimensional toric manifolds are in correspondence with Delzant polygons. These manifolds are $S^2 \times S^2$ and $\mathbb{C}P^2 \#_n \overline{\mathbb{C}}P^2$ where the $n \geq 0$ is the number of blow ups. In 4-dimensions, there are two minimal models, namely $S^2 \times S^2$ and $\mathbb{C}P^2$. These are minimal models in the sense that all other examples can be obtained by blowing up the minimal models. Alternatively, we can blow down any toric 4-manifold to one of these minimal models.

Recently, in 2023, Pelayo and Santos showed that in 6-dimensions for Delzant polyhedra there are not finitely many minimal models. In other words, there exist polyhedra corresponding to toric 6-folds with arbitrarily many vertices and edges that can't be blown down.

They also studied the moduli space of Delzant polyhedra. For the 2-dimensional polygons corresponding to the 4-dimesnional toric manifolds, the moduli space of Delzant polygons is simply connected. In contrast, for the 3-dimensional polyhedra corresponding to the 6-dimensional toric manifolds, the moduli space is not simply connected.

Rigid group actions on manifolds

Thomas Haettel, IUT Montpellier

The groups we will be considering are lattices $\Gamma \leq \mathrm{SL}(n, \mathbb{R})$, particularly for $n \geq 3$, for example $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. There are natural actions of such Γ on S^{n-1} and T^n, which we will write as $\Gamma \to \mathrm{Diff}(S^{n-1})$ and $\Gamma \to \mathrm{Diff}(T^n)$ to emphasize how Γ is acting, which begs the following question:

Question (Zimmer)**.** Are there other natural actions of lattices on manifolds?

What is known:

Theorem (Brown-Fisher-Hurtado '16, Brown-Damjanović-Zhang '18)**.** $\Gamma \to \mathrm{Diff}(M^d)$ has finite image for M a d-manifold under the hypothesis $d < m - 1$.

Theorem (Ghys '99, Burger-Monod '99)**.** $\Gamma \to \mathrm{Homeo}(S^1)$ has a finite image.

Theorem (Lifschitz-Witte-Morris '04, Deroin-Hurtado '20)**.** $\Gamma \to \mathrm{Homeo}(\mathbb{R})$ has finite image.

What is open:

- Is there a similar result for $\Gamma \to \mathrm{Homeo}(\mathbb{R}^2)$?
- Does there even exist an action of a lattice $\Gamma \to \mathrm{Homeo}(\mathbb{R}^2)$?
- Does there exist an action $\Gamma \to \mathrm{Diff}(\Sigma_g)$ for $g \geq 2$?
- What about analagous results for lattices in $\mathrm{SL}(n, \mathbb{Q}_p)$?
- Does Γ having property (T) guarantee the existence of an action $\Gamma \to \mathrm{Homeo}(\mathbb{R})$?

Uniqueness of splitting spheres

Maggie Miller, Stanford University

Here's a theorem posted to the arXiv a few years ago:

Theorem (Budney–Gabai)**.** There exist embeddings $S^3 \hookrightarrow S^3 \times S^1$ that are homotopic but not isotopic to $S^3 \times \{\text{pt}\}$.

It seems reasonable to think that the above result may be useful in addressing the following question:

Question. Are splitting spheres unique in dimension 4?

Here a splitting sphere is a sphere that splits a link. A link is a collection of (say 2) codimension 2 spheres embedded in (say) S^4. A sphere $S^3 \hookrightarrow S^4$ is said to split a link L if the two components of the link L are in different components of $S^4 \setminus S^3$.

The L-space conjecture for thin knots

Liam Watson, University of British Columbia

Let K be a knot. We use V_K to denote its Jones polynomial, $\widetilde{Kh}$ to denote its reduced Khovanov homology, and χ_δ to denote its Euler characteristic with respect to the ($\delta = q/2 - h$)-grading. We have:

$$\det(K) = V_K(-1) = \chi_\delta(\widetilde{Kh}(K)) \leq \dim \widetilde{Kh}(K).$$

A knot is *thin* if $\widetilde{Kh}$ is supported in only one δ-grading, and for thin knots, the final inequality above is an equality.

A group G is *left-orderable* if there exists $\emptyset \neq \mathcal{P} \subset G$ such that:

- $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{P}$, and
- $G = \mathcal{P} \amalg \{1\} \amalg \mathcal{P}^{-1}$.

(If G is countable, it is equivalent to ask that G acts on the line by homeomorphisms.)

Puzzle: if K is thin then it should not be possible to left-order $\pi_1(\Sigma_K)$, where Σ_K is the 2-fold branched cover of S^3, branched along K.

Why should we expect this? Note that $|H_1(\Sigma_K;\mathbb{Z})| = \det(K)$. Oszváth-Szabó have shown that

$$|H_1(\Sigma_K;\mathbb{Z})| \leq \dim \widehat{HF}(\Sigma_K) \leq \dim \widetilde{Kh}(K),$$

meaning that when K is thin, the branched cover Σ_K satisfies

$$|H_1(\Sigma_K;\mathbb{Z})| = \dim \widehat{HF}(\Sigma_K),$$

the condition defining the class of three-manifolds called *L-spaces*. The *L-space conjecture* posits (among other things) that the fundamental groups of L-spaces are not left-orderable. Thus solving the puzzle amounts to proving part of the L-space conjecture for double branched covers of thin knots.

(Another way to think of the non-left-orderable fundamental groups as an expansion on the class of finite finite groups. It is a measure of group smallness.)

Watanabe's Disproof of the Smale Conjecture in Dimension 4

David Gay, University of Georgia

For any n, we can ask if the inclusion $\mathrm{SO}(n+1) \hookrightarrow \mathrm{Diff}^+(S^n)$ is a homotopy equivalence, where $\mathrm{SO}(n+1)$ acts on S^n in the usual manner (and Diff^+ denotes orientation-preserving diffeomorphisms). This is equivalent to $\mathrm{Diff}^+_\partial(B^n)$ being contractible (where Diff^+_∂ denotes orientation-preserving diffeomorphisms that fix the boundary). In 1959, Smale initiated this line of inquiry by showing that we indeed get a homotopy equivalence when $n = 2$. The case of $n = 3$, which came to be known as the Smale conjecture, was proven by Hatcher in 1983. It has also been shown that this result is false in all dimensions $n \geq 5$, leaving only the case of $n = 4$ open.

In 2018, Watanabe settled this final case by detecting nonzero elements of

$$\mathbb{Q} \otimes \pi_{k-1}\big(\mathrm{Diff}^+_\partial(B^4)\big) \cong \mathbb{Q} \otimes \pi_k\big(B\mathrm{Diff}^+_\partial(B^4)\big)$$

for certain dimensions $k \geq 2$ (e.g. for $k = 2$), which implies that $\mathrm{Diff}^+_\partial(B^4)$ is contractible, and so the inclusion $\mathrm{SO}(5) \hookrightarrow \mathrm{Diff}^+(S^4)$ is not a homotopy equivalence. Watanabe utilized Kontsevich characteristic classes on B^4-bundles over S^k that are valued in a $\mathbb{Q}$-vector space $\mathscr{A}_k$ generated by trivalent graphs, in order to define a morphism

$$\mathbb{Q} \otimes \pi_k\big(B\mathrm{Diff}^+_\partial(B^4)\big) \to \mathscr{A}_k.$$

These characteristic classes are described in terms of configuration space integrals, which are quite difficult to compute in practice. But Watanabe was able to pursue more computations by a change in perspective, akin to viewing the cup product in terms of intersections instead of integration of wedged forms. To prove surjectivity, he started with a directed trivalent graph having no sinks or sources, and used the combinatorial data of the graph as gluing instructions to build the desired bundle out of handles (motivated by clasper surgery).

The zeroth homotopy group $\pi_0\big(\mathrm{Diff}^+_\partial(B^4)\big)$ is isomorphic to the smooth (oriented) mapping class group of S^4. Whether this group is trivial remains an open question, but the speaker has an explicit example of a diffeomorphism $S^4 \to S^4$ that *may* not be isotopic to the identity, which can be described by twisting in a neighborhood of a "half-unknotted Montesinos twin" or by composing two diffeomorphisms between the double of the Mazur manifold and S^4.