# Cornell Topology Festival 2024
# Panel Discussion

May 04, 2024

*This is a report on the panel discussion of the 59th Topology Festival at Cornell University, which ran from May 3 to May 5, 2024. Each of the speakers was given approximately 5 minutes to outline either a particularly interesting recent result, or an open problem in the field. Summaries of their presentations follow.*

*Reported by: Conan Gillis, Colby Kelln, Chaitanya Tappu, Chase Vogeli*

### A Conjecture of Bergeron–Venkatesh about Covers of Closed, Arithmetic Hyperbolic 3-Manifolds

*Nathan Dunfield, University of Illinois, Urbana-Champaign*

Let $M_0$ be a closed, hyperbolic 3-manifolds. Consider a tower of finite sheeted regular covers

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

that is *exhausting*, i.e. $\operatorname{systole}(M_k) \to \infty$ or equivalently $\bigcap \pi_1(M_k) = \{e\}$. You can think of this as the tower "unwinding all the topology of $M_0$".

**Conjecture.** *If $M_0$ is arithmetic and $M_k$ is "congruence", and we have an exhausting tower of covers of $M_0$ by $(M_k)$, then as $k \to \infty$,*

$$\frac{\log |\mathrm{H}_i(M_k;\mathbb{Z})_{\text{torsion}}|}{\mathrm{Vol}(M_k)} \to \frac{1}{6\pi}.$$

Since first homology is the abelianization of the fundamental group, this conjecture is equivalent to the following:

**Conjecture** (restatement of above). *If $M_0$ is arithmetic and $M_k$ is "congruence", and we have an exhausting tower of covers of $M_0$ by $(M_k)$, then as $k \to \infty$,*

$$\frac{\log |\mathrm{ab}(\pi_1(M_k))_{\text{torsion}}|}{[\pi_1(M_0) : \pi_1(M_k)]} \to \frac{\mathrm{Vol}(M_0)}{6\pi}.$$

Note that if this conjecture is true, then this conjecture implies that volume is a profinite invariant for arithmetic hyperbolic 3-manifolds.

### Quasiisometric invariance of coherence

*Kasia Jankiewicz, University of California, Santa Cruz*

**Theorem** (Jaikin-Zapivain–Linton)**.** *Every one relator group is coherent.*

Here, a finitely generated group is a *one relator group* if it has a finite presentation with one relator, and is *coherent* if each of its finitely generated subgroups is finitely presented. The above theorem settles a conjecture of Baumslag.

Examples of coherent groups are free groups, surface groups (subsumed by the theorem above), and 3-manifold groups. A non example is the product of two nonabelian free groups.

**Question.** Do there exist quasiisometric groups where one is coherent and the other is not? In particular, is there a coherent group quasiisometric to a product of two nonabelian free groups?

### Subgroups of Hyperbolic Groups

*Noel Brady, Oklahoma University*

We have recently found a 4-dimensional group $H$ with finite $K(\pi, 1)$ space which is not hyperbolic, but satisfies the short exact sequence

$$1 \to H \to G \to \mathbb{Z} \to 1$$

for some hyperbolic group $G$. That $H$ embeds into a hyperbolic group implies that $H$ contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ or $BS(m, n)$ for any integers $m, n \geq 1$. Thus, one may ask, are there 2- and 3-dimensional groups $H$ which have finite $K(\pi, 1)$ space, are not hyperbolic, and contain no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ or $BS(m, n)$? Ignoring dimension, it is also an open problem to classify these groups up to quasi-isometry. Finally, what can we say about $Out(H)$?

As a final question, what about Dehn Functions about finitely presented or type F subgroups of hyperbolic groups? Are they $\simeq e^n$? Greater than $e^n$?

### The Polygonal Peg Problem

*Ina Petkova, Dartmouth College*

**Question** (The polygonal peg problem)**.** Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve, and $P \subset \mathbb{R}^2$ be a polygon. Can we inscribe a (similar copy of) $P$ in $\gamma$?

If $P$ is a triangle, the answer is yes, as is easily seen.

If $P$ has at least five sides, the answer is no in general, since two ellipses that are not similar cannot inscribe similar pentagons.

So from now on, we assume that the number of sides is 4, and denote the quadrilateral by $Q$.

**Conjecture** (Toeplitz, 1912/square peg problem)**.** *Does every Jordan curve inscribe a square? This is the polygonal problem specialised to the case where $P$ is a square.*

The polygonal peg problem has been solved under some additional hypotheses on $\gamma$ and $Q$.

| Year | $\gamma$ | $Q$ |
|---|---|---|
| 1913 | smooth convex | square |
| 1929 | smooth (not necessarily convex) | square |
| 1981 | (no additional hypothesis) | (exists) rectangle |
| 2018 | smooth | rectangle, aspect ratio $\sqrt{3}$ |
| 2020 | smooth | rectangle (any aspect ratio) |
| 2024 (two weeks ago!) | smooth | cyclic |

The recent results have used techniques from Lagrangian submanifold theory.

## Coherence / Subgroups of Hyperbolic Groups

*Martin Bridson, University of Oxford*

Sometimes in mathematics, we can prove a lot about highly abstract structures, but know surprising little about more concrete objects. For example, two open questions of Serre ask: (1) Is $SL(3, \mathbb{Z})$ coherent? (2) Is $SL(2, \mathbb{Z}[1/p])$ coherent?

Taking a different tack, we wish to find subgroups $K$ of a finitely generated hyperbolic group $\Gamma$ such that $K$ has some finiteness properties (and inherits the property of having no Baumslag-Solitar subgroups from $\Gamma$) but is not itself hyperbolic, and which can be put in the short exact sequence

$$1 \to K \to \Gamma \to \mathbb{Z} \to 1.$$

A variation on this is to find $K$ as above such that $K$ is type $\mathcal{F}_n$ but not $\mathcal{F}_{n+1}$. A recent result of Llosa-Isenrich-Py constructs groups that satisfy precisely these conditions. In particular, they produce a large class of mutually noncommensurable lattices $\Gamma$ in complex hyperbolic space, i.e. lattices in $\mathrm{PU}(n, 1)$, such that on passing to a sufficiently deep finite index subgroup $\Gamma_0 < \Gamma$, there are lots of homomorphisms $\Gamma_1 \to \mathbb{Z}$ whose kernels satisfy the required properties. In further work, they exhibit more examples of finitely generated hyperbolic groups $\Gamma$ and homomorphisms $\Gamma \twoheadrightarrow \mathbb{Z}^2$ whose kernels satisfy the above.

Relatedly, Llhosa-Isenrich showed if we have the short exact sequence

$$1 \to K \to \Gamma \to \mathbb{Z} \to 1,$$

with $K$ finitely presented and $\Gamma$ hyperbolic, then the Dehn function of $K$ is polynomially bounded. A question one could ask along these lines is: does every finitely presented subgroup of a product $\Gamma_1 \times \cdots \times \Gamma_n$, where each $\Gamma_i$ is a free or surface group, have a polynomially bounded Dehn Function?

## Commutativity in double semigroups

*Eugenia Cheng, School of the Art Institute of Chicago and City, University of London*

A *double monoid* is a set $S$ with two unital associative binary operations $*$ and $\circ$ which satisfy the *interchange law*:

$$(x * y) \circ (z * w) = (x \circ z) * (y \circ w) \text{ for all } x, y, z, w \in S.$$

With interchange, we may depict such double products in a well-defined way as

$$\begin{array}{|c|c|}\hline x & y \\ \hline z & w \\ \hline\end{array}$$

where $*$ is horizontal multiplication and $\circ$ is vertical multiplication. The *Eckmann–Hilton argument* proceeds by four applications of the unit law

$$\begin{array}{|c|c|}\hline x & y \\ \hline\end{array} = \begin{array}{|c|c|}\hline x & 1 \\ \hline 1 & y \\ \hline\end{array} = \begin{array}{|c|}\hline x \\ \hline y \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & x \\ \hline y & 1 \\ \hline\end{array} = \begin{array}{|c|c|}\hline y & x \\ \hline\end{array}$$

which show that the operations $*$ and $\circ$ coincide and are commutative. This argument originated from the study of higher homotopy groups of topological spaces but now finds applications to higher categories. For instance, it shows that doubly degenerate higher categories exhibit commutativity phenomena.

In his paper, "Note on commutativity in double semigroups and two-fold monoidal categories," Joachim Kock discovered that it is still possible to show certain commutativity phenomena without the presence of units. Specifically, twelve applications of the associative law show that

$$\begin{array}{|c|c|c|c|}\hline \phantom{x} & & & \\ \hline & x & y & \\ \hline & & & \\ \hline & & & \\ \hline\end{array} = \begin{array}{|c|c|c|c|}\hline \phantom{x} & & & \\ \hline & y & x & \\ \hline & & & \\ \hline & & & \\ \hline\end{array}$$

in any *double semigroup*, i.e. set with two associative non-unital multiplications satisfying interchange. The blank boxes represent elements that stay fixed on both sides of the equation.

### Isoperimetric inequalities with lattice point capacities

*Morgan Weiler, Cornell University*

Let $\mathbb{Z}^2$ be the standard integer lattice in the plane $\mathbb{R}^2$. For a convex lattice polygon $\Omega$ in the plane containing the origin, define an associated dual norm using the formula $\|v\|^* = \max\{\langle v, w\rangle \mid w \in \partial\Omega\}$. For a polygon $\Lambda$, define its $\Omega$-*length* to be $l_\Omega(\Lambda) = \sum_{v \text{ edge of } \Lambda} \|v\|^*$. The following theorem may be viewed as the solution to the isometric problem in the dual norm.

**Theorem** (Wulff, 1901)**.** *The unique minimiser of* $l_\Omega(\Lambda)$ *over polygons* $\Lambda$ *with* $\operatorname{area}(\Lambda) = \operatorname{area}(\Omega)$ *is* $\Omega$ *itself.*

However, there is a well known relationship between the area of a lattice polygon and the number of lattice points it contains.

**Theorem** (Pick's theorem)**.** *Let $P$ be a lattice polygon in the plane, $I$ be the number of lattice points in its interior and $B$ be the number of lattice points on its boundary. Then* $\text{area}(P) = I + \frac{1}{2}B - 1$.

This leads to a question: Does Wulff's theorem hold if the number of lattice points in a polygonal region is used as a proxy for the area of the polygon?

**Question.** Is $\Omega$ the unique minimiser of $l_\Omega(\Lambda)$ over lattice polygons $\Lambda$ satisfying $|\Lambda \cap \mathbb{Z}^2| = |\Omega \cap \mathbb{Z}^2|$?

## Unnatural periodic orbits force infinitely many periodic orbits

*Victor Ginzburg, University of California, Santa Cruz*

2 is the lower bound, it is the minimum possible number of fixed points of an area preserving diffeomorphism of the sphere.

**Conjecture.** *A Hamiltonian diffeomorphism of the $n$-dimensional projective space with at least $n+1$ fixed points has infinitely many periodic orbits.*

Overall the theme is that every Hamiltonian system, every phase space, has a threshold. Whenever the system has more periodic orbits than this threshold, it has infinitely many periodic orbits.

Recently for $\mathbb{C}\mathrm{P}^n$ and some other small class of Hamiltonian systems, the conjecture was proved but with very minor additional hypotheses.

But there is a slightly different interpretation of what is going on. Rather than simply counting the number of periodic orbits, we need to actually look at their nature. If there is a periodic orbit which does not have to be there, which looks out of place, then there are infinitely many periodic orbits. This theme has many variations, and can be specialised to various phase spaces.

Here are some examples, and there are probably tons of others. Take a Hamiltonian diffeomorphism, say of a torus, or more generally of a surface or of any compact manifold with a significant fundamental group. Note that a Hamiltonian diffeomorphism need not have any noncontractible periodic orbits (which is a dramatic difference from the special case of geodesic flow). However, if it has at least one noncontractible periodic orbit, then it has infinitely many noncontractible periodic orbits which are genuinely distinct.

One can also go in the opposite direction by looking at the geodesic flow on a Riemannian manifold, say a torus or a higher genus surface. Then you can have metrics without contractible periodic orbits, that is, contractible closed geodesics. Now assume that there is a metric which does have a contractible geodesic, that is, there is a periodic orbit which does not have to be there. Then there are necessarily infinitely many prime contractible geodesics.

## A Criterion for arithmeticity of a Lattice in $\mathrm{Isom}^+(\mathbb{H}^n)$

*Bruno Martelli, Università di Pisa*

Here is a very exciting theorem.

**Theorem** (Bader-Fisher-Miller-Stover)**.** *Let* $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^n)$ *be discrete and let the quotient* $\mathbb{H}^n/\Gamma$ *have finite volume (in other words, let* $\Gamma$ *be a lattice). If* $\mathbb{H}^n/\Gamma$ *contains infinitely many finite volume totally geodesic immersed submanifolds of dimension at least 2, then* $\Gamma$ *is arithmetic.*

It is okay if you do not know what it means for $\Gamma$ to be arithmetic. Just appreciate the beauty of the following corollary.

**Corollary.** *The figure eight knot is the only knot whose complement is hyperbolic and whose complement in* $S^3$ *contains infinitely many finite volume totally geodesic immersed surfaces.*