

Cornell Topology Festival 2025

Panel Discussion

May 03, 2025

This is a report on the panel discussion of the 60th Topology Festival at Cornell University, which ran from May 2 to May 4, 2025. Each of the speakers was given approximately 5 minutes to outline either a particularly interesting recent result, or an open problem in the field. Summaries of their presentations follow.

Reported by: Isaac Broudy, Chase Vogeli, Colby Kelln, Conan Gillis, Kimball Strong

Cubulating manifolds

Jeremy Kahn, Brown University

Theorem. *If M is a compact 3-dimensional manifold with $\pi_2(M) = 0$, then M has finite cover $\hat{M}$, with an embedded π_1 -injective, non- S^2 , orientable surface.*

By the Thurston-Hamilton-Perelman Geometrization of 3-manifolds We can cut M along π_1 -injectively embedded surfaces, into geometric pieces.

If M is a closed hyperbolic 3-manifold, we have $M \cong \mathbb{H}^3/\Gamma$ for $\Gamma < PSL_2(\mathbb{C})$ a lattice. We can cubulate M using surface subgroups and cut M into pieces using using cube complexes.

Question. Can we cubulate another, more general class of manifolds? Can we cubulate hyperbolic 4-manifolds?

Finitely generated and torsion-free groups acting on surfaces

Kathryn Mann, Cornell University

Question. Is there an example of Γ finitely generated and torsion-free, which is not the subgroup of $\text{Homeo}(S)$, for some surface S ?

$\text{Homeo}(S)$ admits an action on the curve graph of S . We can understand the structure of $\text{Homeo}(S)$ via the dynamics of the action.

Definition. The curve graph of a surface, S , is a graph whose vertices are essential simple closed curves in S (not taken up to isotopy) and edges defined by the relation of disjointness of two curves. We denote the curve graph of S , $C^+(S)$

Theorem. *Let S_g be a surface of genus g . For $g \geq 1$, $C^+(S_g)$ is Gromov-hyperbolic.*

Hyperbolicity in non-finitely generated groups

Nicholas Vlamis, City University of New York

Fact. *Hyperbolic groups are nice and people like them. For evidence of niceness, hyperbolic groups are finitely presented (and in fact are of type F_∞), and have solvable word problem.*

Question. Can the notion of some type of hyperbolicity be extended to non-finitely generated groups?

To define a hyperbolic group, one first restricts to considering finitely generated groups (if a finitely generated group has a finite generating set with respect to which the Cayley graph is δ -hyperbolic in the sense of Gromov, then we call the group a *hyperbolic group*). We could weaken the assumption on our group of finite generation to, say, compact generation, and the requirement on the generating set realizing hyperbolicity of the word metric to being a compact generating set. The paper with the following result does this and characterizes groups that are amenable and hyperbolic in this generalized sense.

Theorem (Caprace-Cornuier-Monod-Tessera <https://arxiv.org/abs/1202.3585>). *Let G be a locally compact and compactly generated topological group. G is non-elementary hyperbolic and amenable if and only if $G \cong H \rtimes_\alpha \mathbb{Z}$ or $G \cong H \rtimes_\alpha \mathbb{R}$ where $\alpha(1)$ is “compacting” and H is noncompact.*

Some context or analogy in the non-generalized hyperbolic group setting for this result can come from considering the hyperbolic plane as the upper half space model. The stabilizer of infinity inside the group of isometries of $\mathbb{H}^2$ is a non-elementary hyperbolic group that is amenable and isomorphic to $\mathbb{R} \rtimes \mathbb{R}$ where the first factor comes from translation to the left or right and the second factor comes from upward hyperbolic isometries. More generally, a finitely generated hyperbolic group is amenable if and only if it is isomorphic to $\mathbb{Z}$, so moving into the world of compactly generated hyperbolicity makes the subclass of amenable groups richer.

Geodesic Representatives of Isotopy Classes on Hyperbolic Manifolds

Macarena Arenas, Cambridge University

The following is well-known (cf. Do Carmo’s textbook)

Theorem Given a hyperbolic (or, even, Riemannian) surface M , every free homotopy class contains a minimally self-intersecting geodesic representative. $\square$

Theorem (Scott + Agol) For some M as above, not every isotopy class has such a geodesic representative.

Proof: Uses, among many other things, the Gauss-Bonnet theorem.

Manifolds and Probably Approximately Correct Learning

Damian Osajda, University of Wroclaw

We explore connections between PAC (Probably Approximately Correct) Learning Schemes and Geometric Group Theory.

Definition (Valiant, 1984) $H \subseteq Y^X$ is PAC learnable if there exists a special type of machine learning (i.e. probabilistic) algorithm corresponding to H .

Theorem (Brukhim, Carmon, Dinur, Moran, Yahudayoff, 2024) H is PAC learnable iff the DS dimension is finite. However, there exist PAC Learnable H 's with infinite Natarajan dimension.

Proof: This uses the construction of a finite flag non-square simple pseudomanifold of large dimension.

Telescopic purity for algebraic K -theory

Liam Keenan, Brown University

Work of Land–Mathew–Meier–Tamme, building on prior work of Clausen–Mathew–Naumann–Noel, establishes the following “purity” theorem for telescopically localized algebraic K -theory.

Theorem. *For R and ring spectrum and $n \geq 0$, the map*

$$L_{T(n+1)}K(R) \rightarrow L_{T(n+1)}K(L_{T(n) \oplus T(n+1)}R)$$

is an equivalence.

The telescopic localization functors $L_{T(n)}$ can be thought of as breaking a spectrum into its “pieces,” and the effect of this theorem is that $T(n)$ -local K -theory only sees the $T(n) \oplus T(n+1)$ -local piece.

An analogous result holds for the topological cyclic homology of connective ring spectra.

Theorem. *For $n \geq 1$, if a map $A \rightarrow B$ of connective ring spectra is a $T(n) \oplus T(n+1)$ -local equivalence, then the induced map*

$$L_{T(n+1)}\mathrm{TC}(A) \rightarrow L_{T(n+1)}\mathrm{TC}(B)$$

is an equivalence.

The proof of this theorem relies on comparing K -theory and topological cyclic homology via the cyclotomic trace map $K \rightarrow \mathrm{TC}$.

Question. Is it possible to prove a purity result for TC without using K -theory?

New foundations for equivariant cohomology theories

Tyler Lawson, University of Minnesota

For G a finite group, there are several interesting kinds of G -equivariant cohomology in practice: Borel cohomology, Bredon cohomology, Tate cohomology, etc. Whereas ordinary cohomology theories are represented by spectra, equivariant cohomology theories are represented by objects known as G -spectra.

The issue is that the original construction of the G -stable homotopy category require keeping track lots of data. For example, one starts by fixing a G -universe: a certain infinite-dimensional real inner product space with a G -action by isometries. Different choices of universe yield categories of G -spectra with different kinds of algebraic operations (called *transfers*) in a way that is not readily apparent.

Recently, Clark Barwick along with his collaborators has set up new foundations for G -spectra that are more efficient and flexible. One chooses directly which kinds of transfers to encode and considers functors from a category parameterizing these transfers to spectra. This formalism not only handles that case of equivariant stable homotopy theory, but also motivic stable homotopy theory.

Bounded p -power torsion in unstable homotopy groups

Ishan Levy, University of Copenhagen

Unstable homotopy groups of spheres have some weird, interesting stuff. Here's an example: what's $\pi_k(S^3)$ like, for $k > 3$? Well, it is a finitely generated abelian group, so to understand it we need to understand its rationalization (which is trivial by some classical theorems) and also understand it " p -locally" which is to say understand its localization at every prime p , $\pi_k(S^3)_{(p)}$.

Theorem. *Let p be a prime number greater than 2, and $k > 3$. Then every nonzero element of $\pi_k(S^3)_{(p)}$ has order p .*

[Selick] This is a little bit surprising - in the stable homotopy groups of spheres, there are elements of arbitrarily high prime power order for any prime. Unstably, though, we have bounds. In fact, this generalizes for higher dimensional spheres:

Theorem (Cohen-Moore-Neisendorfer). *Let p be a prime number greater than 2. Then every element of $\pi_k(S^{2n+1})_{(p)}$ has order at most p^n .*

Note that while these are stated for odd spheres, there is an odd-prime equivalence $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$, so the homotopy groups of even spheres are expressible in terms of odd spheres. (There are slightly more complex but similarly-natured results 2-locally). Now, all of this is related to this fascinating dichotomy in rational homotopy theory, which says that a finite dimensional simply-connected CW -complex X must be either:

- *rationally elliptic*, meaning that $\pi_*(X) \otimes \mathbb{Q}$ is zero for sufficiently high dimension (equivalently, is finite dimensional), or
- *rationally hyperbolic*, meaning that the dimensions of the vector spaces $\pi_k(X) \otimes \mathbb{Q}$ grown exponentially in k .

Moore's conjecture, which is still an open problem, asserts a tight connection between the property of being rationally hyperbolic and having large p -power torsion:

Conjecture (Moore). *Let X be a finite dimensional simply connected CW complex. Then the following are equivalent:*

- X is rationally hyperbolic (see above)
- For some prime p , $\pi_*(X)_{(p)}$ has elements of arbitrarily high order (equivalently, no power of p annihilates $\pi_*(X)$)
- For all primes p , $\pi_*(X)_{(p)}$ has elements of arbitrarily high order (equivalently, no natural number annihilates $\pi_*(X)$)

The computations with odd spheres above are an example where this is known to be true: they do not satisfy bullet points 2 or 3; it is also known that the dimension of $\pi_*(X) \otimes \mathbb{Q}$ is 1, so they are not rationally hyperbolic.

The periodicity theorem in chromatic homotopy theory

Irina Bobkova, Texas A&M University

In chromatic homotopy theory, the Morava K -theories $K(n)$ are a collection of homology theories with coefficient rings given by

$$K(n)_* \cong \mathbb{F}_p[v_n^\pm], \quad |v_n| = 2p^n - 2.$$

The *periodicity theorem* of Hopkins–Ravenel–Smith states that these homology theories detect periodic phenomena in a manner we now describe.

In what follows, let X be a finite CW complex. One can show that if the reduced $K(n)$ -homology of X vanishes, then so does its reduced $K(n-1)$ -homology. We say X has *type n* if n is the smallest integer such that X has nonzero $K(n)$ -homology.

Theorem. *If X has type n , then there exists a map $f : \Sigma^d X \rightarrow X$ for some $d \geq 0$ such that f induces an isomorphism*

$$K(n)_*(\Sigma^d X) \xrightarrow{\cong} K(n)_*(X).$$

The self-maps furnished by the periodicity theorem can be used to construct interesting elements in the stable homotopy groups of spheres $\pi_*\mathbb{S}$. If X has cells ranging in dimensions from 0 to e , a self map $f : \Sigma^d X \rightarrow X$ fits into a composite

$$S^d \rightarrow \Sigma^d X \xrightarrow{f} X \rightarrow S^e$$

where the first map is the inclusion of the bottom d -dimensional cell of $\Sigma^d X$, and the last map quotients X outside of an e -dimensional cell. This composite determines a class in $\pi_{d-e}\mathbb{S}$.